REAL ALGEBRAIC GEOMETRY LECTURE NOTES (24: 09/07/15)

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Contents

The aim of this lecture is to give a proof of the so called Baer-Krull Representation Theorem. Moreover, we prove one of its consequences.

1. Preliminaries

Recall 1.1. Let K be a field. A subset $P \subseteq K$ is said to be a **positive cone** of K, if and only if

(i) $P + P \subseteq P$,

(ii) $P \cdot P \subseteq P$,

(iii) $-1 \notin P$,

(iv) $P \cup -P = K$.

Recall 1.2. If P is a positive cone, we can define an ordering $\leqslant_P$ on K by

$$\forall x, y \in K\colon x \leqslant_P y :\Leftrightarrow y - x \in P.$$

Conversely, if $\leqslant$ is an ordering on K, then $P_{\leqslant} := K^{\geqslant 0}$ is a positive cone.

Notation 1.3. Let (K, v) be a valued field. Let Γ be the value group of v. The quotient group $\overline{\Gamma} = \Gamma/2\Gamma$ becomes in a canonical way an $\mathbb{F}_2$-vector space. We denote by $\overline{\gamma} = \gamma + 2\Gamma$ the residue class of $\gamma \in \Gamma$.
Let $\{\pi_i : i \in I\} \subseteq K^*$ such that $\{\overline{v(\pi_i)} : i \in I\}$ is an $\mathbb{F}_2$-basis of $\overline{\Gamma}$. Then $\{\pi_i : i \in I\}$ is called a **quadratic system of representatives** of K with respect to v.
Moreover, we say a valuation ring is P-convex, iff it is $\leqslant_P$-convex.

2. Baer-Krull Representation Theorem

Theorem 2.1. *(Baer-Krull Representation Theorem)*
Let (K,v) be a valued field. Let $\mathcal{X}(K)$ and $\mathcal{X}(Kv)$ denote the set of all orderings (respectively positive cones) of K and Kv, respectively. Fix some quadratic system $\{\pi_i : i \in I\}$ of representatives of K with respect to v.
Then there is a bijective correspondence

$$\{P \in \mathcal{X}(K) : K_v \text{ is } P\text{-convex}\} \longleftrightarrow \{-1,1\}^I \times \mathcal{X}(Kv)$$

described as follows: Given a positive cone P on K such that K_v is P-convex, let $\eta_P : I \to \{-1,1\}$, where $\eta_P(i) = 1 \Leftrightarrow \pi_i \in P$. Then the map

$$P \mapsto (\eta_P, \overline{P})$$

is the above bijective correspondence.

Proof. Given a mapping $\eta : I \to \{-1,1\}$ and a positive cone Q on Kv, we will define a a positive cone $P(\eta,Q)$ of K, such that K_v is $P(\eta,Q)$-convex and $P(\eta,Q)$ is mapped to (η,Q) by the (bijective) correspondence above.
Let $a \in K^*$. As $\{\overline{v(\pi_i)} : i \in I\}$ is a basis of $\Gamma/2\Gamma$, there exist uniquely determined indices $i_1,\dots,i_r$ such that

$$\overline{v(a)} = \overline{v(\pi_{i_1})} + \ldots + \overline{v(\pi_{i_r})}.$$

Thus, for some $b \in K$, one has

$$v(a) = v(\pi_{i_1}) + \ldots + v(\pi_{i_r}) + 2v(b).$$

Hence, we find some $u \in U_v$ such that

$$a = u\pi_{i_1}\cdots\pi_{i_r}b^2.$$

Note that since b is determined up to a unit, u is determined up to a unit square. Let $\eta : I \to \{-1,1\}$ be a mapping and $Q \in \mathcal{X}(Kv)$ a positive cone on Kv. We define $P(\eta,Q) \subset K$ by $0 \in P(\eta,Q)$ and for each $a \in K^*$ with $a = u\pi_{i_1}\cdot\ldots\cdot\pi_{i_r}b^2$ as above,

$$a \in P(\eta,Q) :\Leftrightarrow \eta(i_1)\cdots\eta(i_r)\overline{u} \in Q.$$

Note that $P(\eta,Q)$ is well-defined as u and hence $\overline{u}$ is determined up to a unit square and $i_1,\dots,i_r$ are completely determined. We have to show that $P(\eta,Q)$ is a positive cone and that K_v is $P(\eta,Q)$-convex.
Let $a,a' \in P(\eta,Q)$ with $a,a' \neq 0$. Moreover, let $u,u' \in U_v$, $b,b' \in K$ and $i_1,\dots,i_r,j_1,\dots,j_s \in I$ such that

$$a = u\pi_{i_1}\cdots\pi_{i_r}b^2,$$
$$a' = u'\pi_{j_1}\cdots\pi_{j_s}(b')^2.$$

If $v(a) \neq v(a')$, say $v(a) < v(a')$, then $v(a+a') = v(a)$. Hence, $a+a' = ca$ for some $c \in U_v$. Note that $\frac{a'}{a} \in I_v$. Thus, from $1 + \frac{a'}{a} = c$ follows $\overline{c} = \overline{1}$. We obtain $a + a' = cu\pi_{i_1}\cdots\pi_{i_r}b^2$.
As $a \in P(\eta,Q)$ we have

$$\begin{aligned} Q \ni \eta(i_1)\cdots\eta(i_r)\overline{u} &= \eta(i_1)\cdots\eta(i_r)\overline{1}\overline{u} \\ &= \eta(i_1)\cdots\eta(i_r)\overline{cu}. \end{aligned}$$

Hence $a + a' \in P(\eta, Q)$.
If $v(a) = v(a')$, then $\{i_1, \ldots, i_r\} = \{j_1, \ldots, j_s\}$. Further $b' = bu''$ for some $u'' \in U_v$. Hence

$$a + a' = (u + u'(u'')^2)b^2\pi_{i_1} \cdots \pi_{i_r}.$$

If $\eta(\pi_{i_1}) \cdots \eta(\pi_{i_r}) = 1$, then $\overline{u}, \overline{u'} \in Q$ and hence

$$\overline{u + u' + (u'')^2} = \eta(\pi_{i_1}) \cdots \eta(\pi_{i_r})\overline{u + u' + (u'')^2} \in Q,$$

i.e. $a + a' \in P(\eta, Q)$.
If $\eta(\pi_{i_1}) \cdots \eta(\pi_{i_r}) = -1$, then $-\overline{u}, -\overline{u'} \in Q$. Hence

$$-\overline{u + u'(u'')^2} = \eta(\pi_{i_1}) \cdots \eta(\pi_{i_r})\overline{u + u'(u'')^2} \in Q,$$

and therefore $a + a' \in P(\eta, Q)$.

In order to prove that $P(\eta, Q)$ is closed under multiplication, we extend η to an $\mathbb{F}_2$-linear map from $\overline{\Gamma}$ to $\{-1, 1\}$. We define $\eta(\overline{v(\pi_i)}) = \eta(i)$, which determines the map completely. By composing

$$K^* \xrightarrow{v} \Gamma \to \overline{\Gamma} \xrightarrow{\eta} \{-1, 1\}$$

we obtain a group homomorphism $K^* \to \{-1, 1\}$, which we also denote by η. We have $a \in P(\eta, Q)$ if and only if $\eta(a)\overline{u} \in Q$ for all $a \in K^*$. From this it follows at once that $aa' \in P(\eta, Q)$ for $a, a' \in P(\eta, Q)$.

As $Q \cup -Q = Kv$, it is clear from the definition that

$$P(\eta, Q) \cup -P(\eta, Q) = K$$

and as $-1 \notin Q$, it is clear that $-1 \notin P(\eta, Q)$.
Further note that $1 + I_v \subseteq P(\eta, Q)$. Thus, K_v is $P(\eta, Q)$-convex. This shows that $P(\eta, Q)$ is a positive cone of K and that K_v is $P(\eta, Q)$-convex.

Let $u \in U_v \cap P(\eta, Q)$. Then it follows from the definition that $\overline{u} \in Q$. Hence $\overline{P(\eta, Q)} \subseteq Q$. As $\overline{P(\eta, Q)}$ and Q are both positive cones, $\overline{P(\eta, Q)} = Q$. Moreover, $\pi_i \in P(\eta, Q) \Leftrightarrow \eta(\pi_i) = 1$ is clear from the definition. This proves surjectivity of the map described in the claim.
Injectivity: Assume P is mapped to (η, Q). It is clear from the definition that $P(\eta, Q) \subseteq P$ and hence P and $P(\eta, Q)$ are both positive cones, i.e. $P(\eta, Q) = P$.

□

Remark 2.2. $\Theta(\leqslant) := \{x \in K : x, -x \leqslant a \text{ for some } a \in \mathbb{Z}\}$, called the **$\leqslant$-convex hull of** $\mathbb{Z}$, defines a valuation ring on K which is non-trivial iff $\leqslant$ is non-Archimedean.

Corollary 2.3. *The field K admits a non-trivial ordering iff K carries a non-trivial valuation with real residue field.*

Proof. Assume that $\leqslant$ is a non-Archimedean ordering on K. Then $\Theta(\leqslant)$ is non-trivial and $\leqslant$-convex. By Baer-Krull, $P_\leqslant$ corresponds to $(\eta_{P_\leqslant}, \overline{P_\leqslant})$. In particular, $\overline{P_\leqslant}$ is a positive cone on Kv, i.e. Kv is real.

Conversely, let $\Theta = \Theta_v$ be a non-trivial valuation ring on K. Let Q be an ordering on its residue field Kv. Choose $\eta = 1$ the constant map. Then there exists an ordering P of K for which Θ is P-convex and $\eta = \eta_P$ and $Q = \overline{P}$. Note that $\Theta(\leqslant) \subseteq \Theta$, hence $\Theta(\leqslant) \neq K$, and therefore the ordering is non-Archimedean. $\square$