REAL ALGEBRAIC GEOMETRY LECTURE NOTES (19: 22/06/15)

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Contents

1. Final segments

As always, let K be an ordered field and let v denote the natural valuation on K with value group $G := v(K^*)$.

Lemma 1.1. *Let G be a totally ordered abelian group and denote by v_G its natural valuation.*

- (i) *If $G_w \neq \{0\}$ is some convex subgroup of G, then $\Gamma_w := v_G(G_w \backslash \{0\})$ is a non-empty final segment of $\Gamma := v_G(G \backslash \{0\})$ (Γ denotes the value set of G)*
- (ii) *Conversely, if Γ_w is a non-empty final segment of Γ, then*
$$G_w := \{g \in G : v_G(g) \in \Gamma_w\} \cup \{0\}$$
is a convex subgroup of G with $\Gamma_w = v_G(G_w)$.

Proof.

- (i) Clearly Γ_w is non-empty since $G_w \neq \{0\}$. Show Γ_w is a final segment. Let $\gamma \in \Gamma_w$ and $\gamma' \in \Gamma$ such that $\gamma < \gamma'$. We want to show that $\gamma' \in \Gamma_w$.
Now $\gamma \in \Gamma_w$, so let $g \in G_w$ such that $\gamma = v_G(g)$ and let $g' \in G$ such that $v_G(g') = \gamma'$. Now $\gamma < \gamma'$ means $g' << g$, i.e. $n|g'| \leqslant |g|$. Therefore $g' \in G_w$ since G_w is convex. Thus, $\gamma' \in \Gamma_w$ as required.
- (ii) ÜA.

□

Definition 1.2. Let $\Gamma \neq \emptyset$ be a totally ordered set. Define
$$\Gamma^{\text{fs}} := \{F : F \neq \emptyset \text{ a final segment of } \Gamma\}.$$

Remark 1.3. The set Γ^{fs} is totally ordered by inclusion. Indeed, given $F_1 \neq \emptyset, F_2 \neq \emptyset$ final segments, either $F_1 \subseteq F_2$ or $F_2 \subseteq F_1$ (verify!). So Γ^{fs} is a totally ordered set.

Example 1.4.

- For $\Gamma = \mathbb{R}$, what is the order type of Γ^{fs}?
  A non-empty final segment of $\mathbb{R}$ is either of the form $r^+ := [r, \infty)$ or $r^- := (r, \infty)$ for $r \in \mathbb{R}$ (Recall the Dedekind completeness of the reals, see RAG I). Hence
  $$\Gamma^{\text{fs}} = \{r^\pm : r \in \mathbb{R}\}$$
  Clearly $r^- < r^+$. Let $r_1 \neq r_2$, say $r_1 < r_2$. Then $r_2^- < r_2^+ < r_1^- < r_2^-$, i.e. Γ^{fs} is a double covering of $\mathbb{R}$,
  $$\left(\sum_{\mathbb{R}} 2\right) + 1 = \mathbb{R} \times_{\text{lex}} 2 + 1.$$
- Suppose $\Gamma = \mathbb{Q}$ and $F := \{q \in \mathbb{Q} : q > \sqrt{2}\}$. Let $\emptyset \neq F$ be a proper final segment of $\mathbb{Q}$.
  - $q \in \mathbb{Q}$, then $F = [q, \infty) =: q^+$ and $F = (q, \infty) =: q^-$ in $\mathbb{Q}$.
  - $r \in \mathbb{R} \backslash \mathbb{Q}$, $F = r^{-1} \cap \mathbb{Q} = r^+ \cap \mathbb{Q}$.

  I claim that these are all the proper non-empty final segments.

- $\mathbb{Z}^{\text{fs}} = \mathbb{Z} + \{1\}$.

Corollary 1.5. *There is a 1 to 1 correspondence*
$$G_w \mapsto v_G(G_w \backslash \{0\}) = \Gamma_w$$
between the rank of G *and* Γ^{fs}*, where* $\Gamma = v_G(G \backslash \{0\})$.

Corollary 1.6. *There is a bijective correspondence*
$$K_w \mapsto G_w = v(U_w) \mapsto \Gamma_w$$
between the rank of K *and* Γ^{fs}.

Lemma 1.7. *The map*
$$\iota : \Gamma \to \Gamma^{fs}, \gamma \mapsto \gamma^+$$
is an order reversing embedding. Its image consists of those final segments which have a smallest element

Notation 1.8. Let us denote by Γ^* the set Γ endowed with the reverse order.

Corollary 1.9. *The map* $\iota : \Gamma^* \hookrightarrow \Gamma^{fs}, \gamma \mapsto \gamma^+$ *is an order preserving embedding.*

Definition 1.10. A final segment which has a smallest element is called a **principal final segment**.

Corollary 1.11. Γ^* *is isomorphic to the chain of principal final segments of* Γ.