REAL ALGEBRAIC GEOMETRY LECTURE NOTES (08: 04/05/15)

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Contents

1. Pseudo-completeness

In the last lecture we showed that pseudo complete implies maximally valued. Today, we prove the converse implication.

Proposition 1.1. *The Hahn product* $(\mathrm{H}_{\gamma\in\Gamma} B(\gamma), v_{\min})$ *is pseudo-complete.*

Proof. Let $\{a_\rho\}_{\rho\in\lambda}$ be pseudo-Cauchy. Recall that $\gamma_\rho = v(a_\rho - a_{\rho+1})$ is a strictly increasing sequence. Define $x \in \mathrm{H}_{\gamma\in\Gamma} B(\gamma)$ by

$$x(\gamma) = \begin{cases} a_\rho(\gamma) & \text{if } \gamma < \gamma_\rho \text{ for some } \rho. \\ 0 & \text{otherwise.} \end{cases}$$

This is well-defined because if $\rho_1 < \rho_2 \in \lambda$, $\gamma < \gamma_{\rho_1}$ and $\gamma < \gamma_{\rho_2}$, then $v(a_{\rho_1} - a_{\rho_2}) = \gamma_{\rho_1}$
and therefore

$$a_{\rho_1}(\gamma) = a_{\rho_2}(\gamma)$$

(note that $v_{\min}(a-b)$ is the first spot where a and b differ).

Now we show that support(x) is well-ordered.
Let $A \subseteq \text{support}(x)$, $A \neq \emptyset$ and $\gamma_0 \in A$. Then $\exists\, \rho$ such that $\gamma_0 < \gamma_\rho$ and $x(\gamma_0) = a_\rho(\gamma_0)$ with $\gamma_0 \in \text{support}(a_\rho)$. Consider

$$A_0 := \{\gamma \in A : \gamma \leqslant \gamma_0\}.$$

Note that since $x(\gamma) = a_\rho(\gamma)$ for $\gamma \leqslant \gamma_0$ it follows that $A_0 \subseteq \text{support}(a_\rho)$ which is well-ordered, so $\min A_0$ exists in A_0 and it is the least element of A.

We conclude by showing that x is a pseudo-limit. By definition of x follows

$$v(x - a_\rho) \geqslant \gamma_\rho = v(a_{\rho+1} - a_\rho) \quad \forall\, \rho \in \lambda.$$

If $v(x - a_\rho) > v(a_\rho - a_{\rho+1})$, then

$$v(x - a_{\rho+1}) = v(x - a_\rho + a_\rho - a_{\rho+1}) = v(a_\rho - a_{\rho+1}) = \gamma_\rho,$$

but on the other hand we have

$$v(x - a_{\rho+1}) \geqslant \gamma_{\rho+1} > \gamma_\rho,$$

a contradiction. □

Corollary 1.2. *Let (V, v) be a valued vector space with $S(V) = [\Gamma, \{B(\gamma), \gamma \in \Gamma\}]$. Then there exists a valuation preserving embedding*

$$(V, v) \hookrightarrow (\mathrm{H}_{\gamma\in\Gamma} B(\gamma), v_{\min})$$

Proof. The picture is the following:

$$\begin{array}{ccc} V_1' = V & \overset{h'}{\hookrightarrow} & \mathrm{H}_{\gamma\in\Gamma} B(\gamma) = V_2' \\ \text{immediate} \Big| & & \Big| \text{immediate} \\ V_1 & \underset{\sim}{\overset{h}{\longrightarrow}} & \coprod_{\gamma\in\Gamma} B(\gamma) = V_2 \end{array}$$

Let $\mathcal{B}$ be a maximal valuation independent set in V and set $V_1 = \langle \mathcal{B} \rangle_Q$. Then V_1 has a valuation basis and therefore h exists and $V|V_1$ is immediate. □

Hilfslemma 1.3. *Let (V_1, v_1) be maximally valued, (V_2, v_2) a valued vector space and $h : V_1 \to V_2$ a valuation preserving isomorphism. Then (V_2, v_2) is maximally valued.*

Proof. Let $S(V_1) = [\Gamma, \{B(\gamma) : \gamma \in \Gamma\}]$. Assume that V_2 is not maximally valued, so $\exists V_2'$ a proper immediate extension. By our main theorem there exists an embedding h' of immediate extensions V_2' into $\mathrm{H}_{\gamma\in\Gamma} B(\gamma)$. This is impossible, since h' cannot be injective. □

Corollary 1.4. *Let (V, v) be a maximally valued vector space. Then it is pseudo complete. In fact*

$$(V, v) \simeq (\mathrm{H}_{\gamma\in\Gamma} B(\gamma), v_{\min}),$$

where $S(V) = [\Gamma, \{B(\gamma) : \gamma \in \Gamma\}]$.

Proof. By the first corollary, the picture is the following

$$\begin{array}{ccc} & & \mathrm{H}\, B(\gamma) \\ & & \Big| \text{immediate} \\ V & \underset{\sim}{\overset{h}{\longrightarrow}} & V_2. \end{array}$$

Since V is maximally valued, it follows from the Hifslemma that V_2 is maximally valued. Therefore the extension $\mathrm{H}\, B(\gamma)|V_2$ is not proper, i.e. $V_2 = \mathrm{H}\, B(\gamma)$. Thus h is surjective, i.e. h is an isomorphism of valued vector spaces $V \to \mathrm{H}_{\gamma\in\Gamma} B(\gamma)$. □