REAL ALGEBRAIC GEOMETRY LECTURE NOTES (07: 30/04/15)

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Contents

1. Pseudo-completeness

Let (V, v) be a valued Q-vector space. We recall that

- (V, v) is **maximally valued** if (V, v) admits no proper immediate extension.

- (V, v) is **pseudo-complete** if every pseudo-convergent sequence in V has a pseudo-limit in V.

Theorem 1.1. *(V, v) is maximally valued if and only if (V, v) is pseudo-complete.*

Today we will prove the implication:

(V, v) pseudo-complete $\Rightarrow$ (V, v) maximally valued.

It follows from the following proposition:

Proposition 1.2. *Let (V, v) be an immediate extension of (V_0, v). Then any element in V which is not in V_0 is a pseudo-limit of a pseudo-Cauchy sequence of elements of V_0, without a pseudo-limit in V_0.*

Note that once the proposition is established we have pseudo complete $\Rightarrow$ maximally valued. If not, assume that (V, v) is not maximally valued. Then there is a proper immediate extension (V', v') of (V, v). Let $y \in V' \backslash V$. By the proposition y is a pseudo-limit of a pseudo-Cauchy sequence in V without pseudo-limit in (V, v), a contradiction.

Proof. (of the proposition)
Let $z \in V \setminus V_0$. Consider the set

$$X = \{v(z - a) : a \in V_0\} \subset \Gamma.$$

Since $z \notin V_0$, $\infty \notin X$. We show that X can not have a maximal element. Otherwise, let $a_0 \in V_0$ be such that $v(z - a_0)$ is maximal in X. By the characterization of immediate extensions, there exists some $a_1 \in V_0$ such that $v((z-a_0)-a_1) > v(z-a_0)$. So $a_0+a_1 \in V_0$ and $v(z-(a_0+a_1)) > v(z-a_0)$, a contradiction. Thus, X has no greatest element.

Select from X a well-ordered cofinal subset $\{\alpha_\rho\}_{\rho\in\lambda}$. Note that $\{\alpha_\rho\}_{\rho\in\lambda}$ has no last element, as λ is a limit ordinal.

For every $\rho \in \lambda$ choose an element $a_\rho \in V_0$ with

$$v(z-a_\rho) = \alpha_\rho.$$

The identity

$$a_\sigma - a_\rho = (z-a_\rho) - (z-a_\sigma)$$

and the inequality

$$v(z-a_\rho) < v(z-a_\sigma) \qquad (\forall\, \rho < \sigma \in \lambda)$$

imply

$$(*) \qquad v(a_\sigma - a_\rho) = v(z-a_\rho).$$

Thus, $\{a_\rho\}_{\rho\in\lambda}$ is pseudo-convergent with z as a pseudo-limit.

Finally suppose that $\{a_\rho\}_{\rho\in\lambda}$ has a further limit $z_1 \in V_0$.

By a result from the last lecture we have

$$v(z-z_1) > v(a_\sigma - a_\rho).$$

Combining this with $(*)$ we get

$$v(z-z_1) > v(z-a_\rho) = \alpha_\rho \qquad \forall\, \rho \in \lambda$$

and this is a contradiction, since $\{\alpha_\rho\}_{\rho\in\lambda}$ is cofinal in X. □

Theorem 1.3. *Suppose that*

(*i*) *V_i and V_i' are Q-valued vector spaces and V_i' is an immediate extension of V_i for $i = 1, 2$.*

(*ii*) *$h : V_1 \to V_2$ is an isomorphism of valued vector spaces.*

(*iii*) *V_2' is pseudo-complete.*

Then there exists an embedding $h' : V_1' \to V_2'$ such that h' extends h. Moreover h' is an isomorphism of valued vector spaces if and only if V_1' is pseudo-complete.

Proof. The picture is the following:

$$\begin{array}{ccc} V_1' & \xhookrightarrow{h'} & V_2' \\ \text{immediate}\ \Big| & & \Big|\ \text{immediate} \\ V_1 & \xrightarrow[\sim]{h} & V_2 \end{array}$$

Consider the collection of triples (M_1, M_2, g), where

$$V_1 \subseteq M_1 \subseteq V_1',$$
$$V_2 \subseteq M_2 \subseteq V_2',$$

and g a valuation preserving isomorphism of M_1 onto M_2 extending h.

This collection is non-empty, because (V_1, V_2, h) belongs to it. Moreover, one can show that every chain has an upper bound (ÜA). Therefore the conditions of Zorn's lemma are satisfied, i.e. there exists a maximal such triple (M_1, M_2, g). We claim that $M_1 = V_1'$. Assume for a contradiction there exists some $y_1 \in V_1' \setminus M_1$.

(**Note:** If $V_0 \subset V_1 \subset V_2$ are extensions of valued vector spaces and $V_2|V_0$ is immediate, then $V_2|V_1$ and $V_1|V_0$ are immediate)

Since V_1' is an immediate extension of M_1, there exists a pseudo-convergent sequence $S = \{a_\rho\}_{\rho\in\lambda}$ of M_1 without a pseudo-limit in M_1, but with a pseudo-limit $y_1 \in V_1'$. Consider $g(S) = \{g(a_\rho)\}_{\rho\in\lambda}$.

(**Facts/ÜA:**

- (i) the image of a pseudo-convergent sequence under a valuation preserving isomorphism is pseudo-convergent.
- (ii) the image of a pseudo-limit of a pseudo-covergent sequence under a valuation preserving isomorphism is a pseudo-limit of the image of the pseudo-convergent sequence.
- (iii) the image of a pseudo-complete vector space under a valuation preserving isomorphism is pseudo-complete.)

Since g is a valuation preserving isomorphism, $g(S)$ is a pseudo-convergent sequence of M_2 without a pseudo-limit in M_2 but with a pseudo-limit $y_2 \in V_2'$, because V_2' is pseudo-complete.

Let $M_i' = \langle M_i, y_i\rangle_Q$ for $i = 1, 2$, and denote by g' the unique Q-vector space isomorphism of the linear space M_1' onto the linear space M_2' extending g such that $g'(y_1) = y_2$.

We show that g' is valuation preserving: let

$$y = x + qy_1 \qquad x \in M_1 \quad (q \in Q \setminus \{0\})$$

be an arbitrary element of $M_1' \setminus M$. The sequence

$$S(y) = \{x + qa_\rho\}_{\rho\in\lambda}$$

is pseudo-convergent in M_1 with pseudo-limit $y \in M_1'$ and 0 is not a pseudo-limit (otherwise $-x/q \in M_1$ would be a pseudo-limit of S).

It follows that (since $y = x+qy_1$ is a pseudo-limit for the sequence $x+qa_\rho$ which does not have 0 as a pseudo-limit)

$$v(y) = \mathrm{Ult}\, S(y)$$

and similarly

$$v(g'(y)) = \mathrm{Ult}\, S(g'(y)),$$

where

$$S(g'(y)) = \{g'(x) + qg'(a_\rho)\}_{\rho\in\lambda}$$

is a pseudo-convergent sequence of M_2 with pseudo-limit $g'(y) \in M_2'$.

Now $g'_{|M_1} = g$ is valuation preserving from M_1 to M_2. So we have

$$\mathrm{Ult}(S(y)) = \mathrm{Ult}(S(g'(y))),$$

hence

$$v(y) = v(g'(y))$$

as required.

Now if h' is onto, then V_1' is pseudo-complete. Conversely, if V_1' is pseudo-complete, then $h'(V_1')$ is also pseudo-complete and hence maximally valued. So the immediate extension $V_2'|h'(V_1')$ cannot be proper, i.e. $h'(V_1') = V_2'$. Thus, h' is onto as claimed. □