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VALUED FIELDS – EXERCISE 9

To be submitted on Wednesday 12.1.2011 by 14:00 in the mailbox.

Definition.

(1) We say that two linear orderings $(I_1, <_1)$ and $(I_2, <_2)$ have the same *order type* if there is an isomorphism between them (i.e. an order preserving map which is injective and surjective).
(2) For a linear order I, the order type of I is the equivalence class of of all linear ordering that have the same order type as I.
(3) An *order type* is an order type of some linear order. The class of all order types is denoted by OT.
(4) The *rank* of an ordered abelian group Γ is the order type of the set of convex subgroups of Γ. Denote this by $\operatorname{rank}(\Gamma)$.

Question 1.

Suppose $(I_1, <)$ and $(I_2, <)$ are linear orderings. Define the linear order $(I_1, <_1) + (I_2, <_2)$ on the set $I_1 \times \{0\} \cup I_2 \times \{1\}$ as follows:

For $a, b \in I_1$, $(a, 0) < (b, 0)$ iff $a <_1 b$. For $a, b \in I_2$, $(a, 1) < (b, 1)$ iff $a <_2 b$. For $a \in I_1, b \in I_2$, $(a, 0) < (b, 1)$.

(1) Show that this is indeed a linear order.
(2) Show that if $(J_1, <_1')$ has the same order type as $(I_1, <_1)$ and $(J_2, <_2)$ has the same order type as $(I_2, <_2)$ then $(J_1, <_1') + (J_2, <_2')$ has the same order type as $(I_1, <_1) + (I_2, <_2)$.
(3) Define addition on order types (i.e. given two order types, $\mathfrak{I}_1, \mathfrak{I}_2 \in \mathsf{OT}$, define $\mathfrak{I}_1 + \mathfrak{I}_2$).
(4) Is addition commutative?
(5) Define multiplication of order types.

Question 2.

Let Γ be an ordered abelian group, and let Δ be a convex subgroup.

(1) Show that there is a unique linear ordering on Γ/Δ such that the canonical map $\Gamma \to \Gamma/\Delta$ is an order preserving homomorphism.
(2) Show that there is a 1-1 correspondence between $\{\Delta_1 \supseteq \Delta \,|\, \Delta_1 \text{ is a convex subgroup of } \Gamma\}$ and the set of convex subgroups of Γ/Δ.
Hint: consider the map taking Δ_1 to Δ_1/Δ.
(3) Show that $\operatorname{rank}(\Gamma) = \operatorname{rank}(\Delta) + \operatorname{rank}(\Gamma/\Delta)$ (see Question 1).

Question 3.

Suppose K is a field. Show that K is real iff -1 is not a finite sum of squares from K (i.e. there are no $a_1, \ldots, a_n \in K$ such that $-1 = \sum a_i^2$). Use the following steps:

(1) Prove right to left.
(2) Define $\mathcal{P}$ to be the set of all pre-positive cones, i.e. the set of all $P \subseteq K$ such that $-1 \notin P$, $P + P \subseteq P$, $P \cdot P \subseteq P$. Show that $\mathcal{P}$ is not empty.
(3) Show that $\mathcal{P}$ satisfies all the conditions of Zorn's lemma and deduce that there is a maximal element P.

(4) Show that P is a positive cone, i.e. that $P \cup -P = K$ (assume not: let $a \in K \setminus (P \cup -P)$, and show that one can add a or $-a$ to P).

Question 4.
Let (K, v) be a valued field with residue field $\bar{K}$. Prove that the following are equivalent:

(1) K is real (i.e. there exists an ordering on K) and the valuation ring K_v is convex (with respect to this ordering).
(2) $\bar{K}$ is real.
(3) For all $a_1, \ldots, a_n \in K$, $v\left(a_1^2 + \ldots + a_n^2\right) = \min\left\{v\left(a_i^2\right) | 1 \leqslant i \leqslant n\right\}$.

Hint: For (2) equivalent with (3) use Question 3.