Prof. Dr. Salma Kuhlmann Dr. Itay Kaplan

VALUED FIELDS – EXERCISE 2

To be submitted on Wednesday 17.11.2010 by 14:00 in the mailbox.

Definition.

(1) A *place* on a field K is a surjective homomorphism $P : K_P \to \Delta$ where K_P is a sub-ring of K and Δ is a field, such that if $x \notin K_P$ then $1/x \in K_P$ and $P(1/x) = 0$.
(2) A place P is called trivial if $K_P = K$.
(3) The rank of a place P is the maximal number n such that there is a chain of prime ideals of K_P. I.e. the maximal number n such that there are prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ such that $\mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq \mathfrak{p}_3 \subseteq \ldots \subseteq \mathfrak{p}_n \subseteq K_P$.

Question 1.
Prove the following:

Theorem. *If A is a Dedekind domain, then every ideal can be written uniquely as a product of prime ideals.*

Hint: First prove existence – suppose I is an ideal, maximal with the property that it is not a product of primes (why does it exist?). Suppose $I \subseteq P$ where P is prime. Then $P'I$ is an ideal of A, which strictly contains I (otherwise $P'I = I$ and then by Question 2 of ex. 1, $P' \subseteq A$). Now use the maximality assumption on I. Then prove uniqueness – if $P_1 \ldots P_r = Q_1 \ldots Q_{r'}$ then $P_1 \supseteq Q_1 \ldots Q_{r'}$ so $P_1 \supseteq Q_i$ for some i, and so $P_1 = Q_i$. Now multiply both sides by P_1'. Continue inductively.

Question 2.
Prove the following:

Theorem. *If A is a Dedekind domain, then the set of fractional ideal is a group under multiplication with the inverse of a fractional ideal $0 \neq I$ being I' and the unit being A. Furthermore it is a free abelian group with the nonzero prime ideals as generators (every element can be written uniquely in the form $P_1^{r_1} \ldots P_n^{r_n}$ for $r_i \in \mathbb{Z}$).*

Hint: for showing that $I'I = A$: First suppose that I is an ideal of A. Obviously $I'I \subseteq A$. If $P_1 \ldots P_r = I$, then $I' \supseteq P_1' \ldots P_r'$ so $I'I \supseteq A$. For a general fractional ideal I, find some a such that aI is an ideal of A by considering the denominators of the generators of I, and note that $(aI)' = a^{-1}I'$.

Question 3.

(1) Show that every PID is a Dedekind domain. Deduce that every PID is a UFD.
Hint for showing that it is integrally closed: Suppose $M \subseteq \text{quot}(A)$ is as Question 2 of ex. 1, (3), and that $(x/y) M \subseteq M$. Show that $M = aA$ for some $0 \neq a \in \text{quot}(A)$, and deduce that $x/y \in A$.

(2) If A is a UFD then for every principle prime ideal $\mathfrak{p}$ of A, there is a non-trivial place $p : A_{\mathfrak{p}} \to \Delta$ for some field extension of A (where $A_{\mathfrak{p}}$ is the localization of A by $\mathfrak{p}$).
(3) If A is a Dedekind domain then for every prime ideal $\mathfrak{p}$ of A, there is a non-trivial place $p : A_{\mathfrak{p}} \to \Delta$ for some field extension of A.
Hint: Note that given x, y, if $(x/y) = P_1 \dots P_r \cdot Q_1' \dots Q_m'$ and $\mathfrak{p}$ is not one of $Q_1, \dots, Q_m$, then $Q_1 \dots Q_r \nsubseteq \mathfrak{p}$ (because $\mathfrak{p} + Q_1 \dots Q_r = A$, and this is because...) and so there is an element $b \in Q_1 \dots Q_r \backslash \mathfrak{p}$, so $(x/y)\, b \subseteq P_1 \dots P_r$, so x/y can be written as a/b where $b \notin \mathfrak{p}$.
(4) Show that the rank of the place p from (3) is 1.
Hint: Show that if q is a prime ideal of $A_{\mathfrak{p}}$ then $q \cap A$ is a prime ideal of A.

Question 4.
Let F be any field. Let $F((t))$ be the field of formal Laurant series over F, namely: $F((t)) = \left\{ \sum_{i=n}^{\infty} a_i t^i \,|n \in \mathbb{Z}, \forall i \geqslant n \,(a_i \in F) \right\}$. You may also think of elements of $F((t))$ as functions $f : \mathbb{Z} \to F$ such that $\text{supp}\,(f) := \{i \in \mathbb{Z} | f(i) \neq 0\} \subseteq (n, \infty)$ for some $n \in \mathbb{Z}$. Think why are these the same thing.
Addition is defined coordinate-wise: given $f, g \in F((t))$, $(f + g)(i) = f(i) + g(i)$.
Multiplication is defined as follows: given $f, g \in F((t))$, $f{\cdot}g\,(i) = \sum_{k \in \mathbb{Z}} f(k)\, g(i - k)$.

(1) Prove that multiplication is well defined and that $F((t))$ is a field.
(2) Show that there is a non-trivial place P on $F((t))$ onto the field F with $K_P = \{f \in F((t)) \,|\text{supp}\,(f) \subseteq \mathbb{N} = \{0, 1, \dots\}\}$.
(3) Show that this place has rank 1.
Hint: note that f is a unit in K_P iff $\text{supp}\,(f) \subseteq \{1, 2, \dots\}$.
(4) Prove that in fact K_P is a PID and conclude that (3) follows from (4) in Question 3.
Hint: If I is an ideal in K_P, let $f \in I$ be chosen so that $\min(\text{supp}\,(f))$ is minimal. Show that $I = (f)$.