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VALUED FIELDS – EXERCISE 14

Definition.

(1) Suppose $(G, +)$ is an abelian group. An element x is called torsion if there exists some $n \in \mathbb{N}$ such that $nx = x + \ldots + x = 0$.
(2) An abelian group G is divisible if and only if for every positive integer n and every g in G, there exists y in G such that $ny = g$.

Question 1.
This question deals with the divisible hull of a group.

(1) Recall Question 1 from Exercise 5. Suppose M is a module over a commutative ring R. Let S be a multiplicative subset of R. Define $S^{-1}M$. Show that it is a module over $S^{-1}R$.
(2) Let G be an abelian group. Think of G as a module over $\mathbb{Z}$. Let $S = \{x \in \mathbb{Z} | x \neq 0\}$. Let $G^d = S^{-1}G$. This is the *divisible hull* of G. Show that it is a $\mathbb{Q}$-vector space.
(3) Let T be the group of all torsion elements in G. Show that the map $G/T \to G^d$ defined by $g + T \mapsto g/1$ is an injective group homomorphism.
(4) Show that G^d is torsion free.
(5) Show that for every group homomorphism $\varphi : G \to G'$ induces naturally a homomorphism (which we shall also call φ) from G/T to G'/T' where T' is the torsion group in G'.
(6) Show that for any φ as above where G' is divisible, there is a unique $\psi : G^d \to G'/T'$ such that $\psi(g/1) = \varphi(g + T)$ for all $g \in G$.
(7) Conclude that if G is torsion free, for instance if G is an ordered group, then $G \subseteq G^d$ and write down the universal property (as in (6)) that G^d has.
(8) Show that if G is ordered by $\leqslant$, then there is a unique order that extends $\leqslant$ that makes G^d into an ordered abelian group.

Question 2.
Let $\mathrm{rr}(G) = \dim_{\mathbb{Q}}(G^d)$.

(1) Show that $\mathrm{rr}(\mathbb{Z}^n) = n$, $\mathrm{rr}(\mathbb{Q}) = 1, \mathrm{rr}(\mathbb{R}) = \infty$.
(2) Show that if T is the torsion group of G then $\mathrm{rr}(G) = \mathrm{rr}(G/T)$.
(3) Show that this rank (the rational rank) is preserved under isomorphism of groups.
(4) Show that $\mathrm{rr}(G)$ equals to the maximal size of a finite $\mathbb{Z}$-independent set $S \subseteq G$ (i.e. if $a_s \in \mathbb{Z}$ for $s \in S$, then $\sum a_s s = 0 \Rightarrow \forall s\,(a_s = 0)$). In particular, any such set is of the same size.
(5) Show that if $H \leqslant G$ then $\mathrm{rr}(G) = \mathrm{rr}(H) + \mathrm{rr}(G/H)$.

Question 3.
Let K be an algebraically closed valued field, L a valued field extension, $L = K(t), t \notin K$. Since any element of $K[t]$ is a product of linear factors, the valuation on L is determined by $\nu(t-a)$ for $a \in K$. Show that one of the following holds:

(1) $\nu(t-a) = \gamma \notin \Gamma(K)$ for some $a \in K$. Show that $\Gamma(L) = \Gamma(K) + \mathbb{Z}\gamma$, $k(L) = k(K)$. (Hint: Use 2.2.3).
(2) $\nu(t-a) \in \Gamma(K)$ for all $a \in K$, and $\nu(t-a)$ takes a maximal value $\nu(b)$ at some $a, b \in K$. Show that $k(L) = k(K)(e)$ where $e = \overline{((t-a)/b)}$. (Hint: show that $e \notin k(K)$, and use 2.2.2 and Exercise 11, Question 3).
(3) $\nu(t-a) \in \Gamma(K)$ for all $a \in K$, and a maximum is not attained. Show that $K(t)$ is an immediate extension. Hint: $0 = \nu((t-a)/b) < \nu((t-a')/b)$ so $\nu((t-a)/b + (a-a')/b) > 0$.

Question 4.
Give a new proof of the dimension inequality using the previous Question: Let L/K be an extension of valued fields. Then $\text{tr.deg}(k(L)/k(K)) + \text{rr}(\Gamma(L)/\Gamma(K)) \leqslant \text{tr.deg}(L/K)$.
Hints:
(1) First show that it is enough to prove it for the case where L is finitely generated (as a field) over K.
(2) Show that it is enough to prove it for the case where $L = K(t)$ for some $t \notin K$. (Hint: use the following fact – $\text{tr.deg}(L_2/L_0) = \text{tr.deg}(L_2/L_1) + \text{tr.deg}(L_1/L_0)$ for fields $L_0 \subseteq L_1 \subseteq L_2$, and use Question 2 (5)).
(3) Show that the inequality holds in the case where t is algebraic over K (Hint: use the fact that $k(L)/k(K)$ is algebraic and that $\Gamma(L)/\Gamma(K)$ is torsion).
(4) Assume t is transcendental. Show that it is enough to prove it for the case where K is algebraically closed (Hint: show that this does not change right hand side, and left hand side can only increase: note that we only added algebraic elements to $k(K)$ and torsion elements to $\Gamma(K)$).
(5) Use Question 3 to finish.