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VALUED FIELDS – EXERCISE 11

To be submitted on Wednesday 26.1.2011 by 14:00 in the mailbox.

Definition.

(1) We say a valued field (K_2, v_2) is an *extension* of (K_1, v_1) when $O_1 = K_1 \cap O_2$. Denote this by $(K_1, v_1) \subseteq (K_2, v_2)$.
(2) Suppose $K \subseteq L$ is a field extension. Let $K_L^{\text{sep}} = \{x \in L \,|x \text{ is seperable over } K\}$.
(3) The *degree of separability* of L over K is $\left[K_L^{\text{sep}} : K\right]$ and it is denoted by $[L : K]_s$.
(4) The *degree of inseparability* of L over K is $\left[L : K_L^{\text{sep}}\right]$ and it is denoted by $[L : K]_i$.

Question 1.
Suppose (K_2, v_2) is an extension of (K_1, v_1). Prove that

(1) $m_1 = m_2 \cap K_1 = m_2 \cap O_1$.
(2) $O_1^\times = O_2^\times \cap K_1$.
(3) The restriction of v_2 to $K_1^\times$ is equivalent to v_1.
(4) Show that (3) is equivalent to the assumption that (K_2, v_2) is an extension of (K_1, v_1).
(5) Show that the residue field k_1 is a subfield of k_2 and the valuation group Γ_1 is a subgroup of Γ_2.
(6) Deduce from what we did in class that if K_2 is a finite extension of K_1, then so is the extension k_2 of k_1.
(7) Show that it is possible that K_2 is a (finite) separable extension of K_1 while k_2 is a purely inseparable extension of k_1.
Hint: choose a prime $p > 0$. Let v be the p-adic valuation on $\mathbb{Q}$. Extend this valuation to a valuation w on $\mathbb{Q}(X)$ such that $w\left(\sum a_i X^i\right) = \min\{v(a_i)\}$. Let $K_1 = \mathbb{Q}(X)$ with w, and let $K_2 = \mathbb{Q}\left(X^{1/p}\right)$ with some extended valuation. Now, $k_1 = \mathbb{F}_p\left(\bar{X}\right)$ and $\bar{X}$ is transcendental (why?) and $k_2 = \mathbb{F}_p\left(\bar{X}^{1/p}\right)$ (why? use the inequality given in class).
(8) Prove that in the example given in the hint, the natural inclusion of groups given in (3), is actually an isomorphism.

Question 2.
Let K be a field of char. $p > 0$. Suppose $f \in K[X]$ is irreducible such that $f = g\left(X^{p^e}\right)$ for some $e \in \mathbb{N}$, and $g' \neq 0$. Let $L = K(\alpha)$ for some root α of f. In class it was shown that in that case $[L : K]_s$ is the degree of f and $[L : K]_i = p^e$, but some of the details were omitted. Write down the full proof.

Question 3.

(1) Suppose (K, v) is an algebraically closed valued field. Show that k is algebraically closed and that Γ is a divisible group (i.e. if $\gamma \in \Gamma$ and $n \in \mathbb{N}$ then

there exists $\gamma' \in \Gamma$ such that $n\gamma' = \gamma$).
Hint: suppose $f \in O[X]$ is a monic polynomial, then all of its roots lies in O.

(2) Is the converse true? i.e. suppose (K, ν) is a valued field with an algebraically closed residue field and a divisible value group, is it necessarily true that K is algebraically closed?
Hint: Let k be algebraically closed of char. 0 . Consider the following ring: its elements are finite sums of the form $\sum_{i \in s} a_i t^i$ where $s \subseteq \mathbb{Q}$ is finite and $a_i \in k$. Multiplication is defined by $t^i t^j = t^{i+j}$. Show that this is a domain, and let K be its field of fractions. Define a valuation on K with value group $\mathbb{Q}$ and residue field k. Show that $1 + t$ does not have a square root.

Question 4.
Suppose (K, ν) is a valued field, and $L \supseteq K$ is a purely inseparable extension. Show that there is a unique extension (up to equivalence) of ν to a valuation w on L such that $(L, w) \supseteq (K, \nu)$.