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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 8

These exercises will be collected Tuesday 15 December in the mailbox number 15 of the Mathematics department.

Theorem 0.1 (Tarski-Seidenberg Principle) *Let $f_i(\underline{T},X) = h_{i,m_i}(\underline{T})X^{m_i} + \cdots + h_{i,0}(\underline{T})$ for $i = 1,\ldots,s$ be a sequence of polynomials in $n+1$ variables with coefficients in $\mathbb{Z}$, where $\underline{T} = (T_1,\ldots,T_n)$. Let ϵ be a function from $\{1,\ldots,s\}$ to $\{-1,0,1\}$. Then there exists a boolean combination $\mathcal{B}(\underline{T})$ (i.e. a finite composition of conjunctions, disjunctions and negations) of polynomial equations and inequalities in the variables $\underline{T}$ with coefficients in $\mathbb{Z}$ such that for every real closed field R and for every $\underline{t} \in R^n$, the system*

$$\begin{cases} Sign f_1(\underline{t},X) & = & \epsilon(1) \\ & \vdots & \\ Sign f_s(\underline{t},X) & = & \epsilon(s) \end{cases}$$

has a solution x in R if and only if $\mathcal{B}(\underline{t})$ holds true in R.

1. Denote $\underline{X} := (X_1,X_2,X_3,X_4)$. The aim of this exercise is to show, in the following cubic equation

$$f(\underline{X}) := X_1^3 + X_2X_1^2 + X_3X_1 + X_4 = 0, \quad (1)$$

how to eliminate the variable X_1. We will adapt to our context the classical **method of Cardano** of resolution of cubic equations in one variable.

(a) Show that the equation (1) is equivalent to

$$g(X,Y,Z) := X^3 + YX + Z = 0. \quad (2)$$

where X, Y and Z depend themselves polynomially on $\underline{X}$.
(*Hint: use the Tschirnhausen transform* $X_1 = X - \dfrac{X_2}{3}$.)

(b) Consider R a real closed field and $R[i]$ (with $i := \sqrt{-1}$) its algebraic closure. Use the changes of unknown

$$X = U + V \text{ and } Y = -3UV$$

to show that, for any $y,z \in R^2$, there exist $u,v \in R[i]$ such that $w_1 := u^3$ and $w_2 := v^3$ are the two solutions with respect to W of the following equation

$$h(W,y,z) := W^2 + zW - \frac{y^3}{27} = 0. \tag{3}$$

(c) Denote $D(Y,Z) := Z^2 + \frac{4}{27}Y^3$ (which is called the **discriminant** for the cubic equation (2)). Consider $y,z \in R^2$. Deduce that

$$\begin{cases} \text{if } D(y,z) > 0, & \text{then (2) has one real solution } X^{(1)} := X^{(1)}(y,z); \\ \text{if } D(y,z) > 0, & \text{then (2) has two real solutions } X^{(l)} := X^{(l)}(y,z),\ l = 1,2; \\ \text{if } D(y,z) < 0, & \text{then (2) has three real solutions } X^{(l)} := X^{(l)}(y,z),\ l = 1,2,3. \end{cases}$$

and give explicit formulas (the so-called **Cardano formulas**) for these solutions. (*Hint: for the case* $D(y,z) < 0$, *we have* $X^{(k)} = j^k u + \overline{j^k u}$ *for* $k = 0,1,2$ *where* $j^k u, \overline{j^k u}$ *are complex conjugate and* $j := \frac{-1+i\sqrt{3}}{2}$ *is the classical third root of 1.*)

(d) Show that there exists a polynomial $\tilde{f}(X_2,X_3,X_4) \in \mathbb{Z}[X_2,X_3,X_4]$ of degree 6 such that for any real closed field R and any $(x_2,x_3,x_4) \in R^3$, the polynomial $f(X_1,x_2,x_3,x_4) \in R[X_1]$ has:
(i) 1 real root if $\tilde{f}(x_2,x_3,x_4) > 0$;
(ii) 2 real roots if $\tilde{f}(x_2,x_3,x_4) = 0$;
(iii) 3 real roots if $\tilde{f}(x_2,x_3,x_4) < 0$.
NB: we do not count multiplicity.

2. In the following exercise we will illustrate the Tarski-Seidenberg principle and its proof. Consider the following polynomials $\in \mathbb{Z}[T,X]$

$$\begin{cases} f_1(T,X) &= TX^2 + (T+1)X + 1 \\ f_2(T,X) &= X^3 - 3T^2X + 2T^3. \end{cases}$$

(a) Compute $g_1(T,X)$, $g_2(T,X)$ respectively, the remainder of the euclidean division of $f_2(T,X)$ by $f_1(T,X)$, $f_2'(T,X)$ (the partial derivative of $f_2(T,X)$ with respect to X) respectively, in $\mathbb{Q}(T)[X]$.

(b) Compute the X-roots of $f_1(T,X)$, $f_2'(T,X)$, $g_1(T,X)$ and $g_2(T,X)$ in terms of T.

(c) As an example, consider a real closed field R and $t \in R$ with $t >> 1$. Denote by $x_1 < x_2 < \ldots < x_5$ the corresponding roots computed in the preceding question, $x_0 := -\infty$, $x_6 := +\infty$, and $I_k :=]x_k,x_{k+1}[$ for $k = 0,\ldots,5$. Then:

(i) compute the sign matrix

$$\text{Sign}_R(f_1(t,X), f_2'(t,X), g_1(t,X), g_2(t,X))$$

(where each row is $\text{Sign}(f(I_0)), \text{Sign}(f(x_1)), \ldots, \text{Sign}(f(x_5)), \text{Sign}(f(I_5))$ for f being the corresponding function).

(ii) deduce the sign matrix

$$\mathrm{Sign}_R(f_1(t,X), f_2(t,X)).$$

(d) Denote $\tilde{f}_1(T,X) := T^2 g_1(T,X)$ (note that multiplication by T^2 is done to clear the denominator of g_1), $\tilde{f}_2(T,X) := g_2(T,X)$, $\tilde{f}_3(T,X) := f_2'(T,X)$ and $\tilde{f}_4(T,X) := f_1(T,X)$. Then resume the preceding method (compute $\tilde{f}_3'$, $\tilde{g}_1, \ldots, \tilde{g}_4$ etc) until we obtain functions $\tilde{f}_j^{(k)}(T,X)$, $j = 1, \ldots, s_k$ for some step k and number s_k of functions, which do not depend on X anymore.

(e) Compute directly the X-roots of $f_2(T,X)$ in terms of T using Cardano formulas, deduce the sign matrix

$$\mathrm{Sign}_R(f_1(t,X), f_2(t,X)).$$

in terms of t, and verify the preceding results.
<u>NB</u>: the case $t = 0$ has to be treated separately.

(f) Conclude that for any real closed field R and any $t \in R$, the resolution of the semi-algebraic system

$$(I) \quad \left\{ \begin{array}{lclc} f_1(t,X) & = & tX^2 + 2tX + 1 & \triangleright_1 \ 0 \\ f_2(t,X) & = & X^3 + 3t^2X + 2t^3 & \triangleright_2 \ 0. \end{array} \right.$$

for some $\triangleright_1, \triangleright_2 \in \{>\ ,\ \geq\ ,\ =\ ,\ \neq\}$ is equivalent to t solution of a semi-algebraic system only in the variable T.