Universität Konstanz
Fachbereich Mathematik und Statistik
Prof. Dr. Salma Kuhlmann
Mitarbeiter: Dr. Mickaël Matusinski
Büroraum F 409
mickael.matusinski@uni-konstanz.de

Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 8

Theorem 0.1 (Tarski-Seidenberg Principle) *Let $f_i(\underline{T},X) = h_{i,m_i}(\underline{T})X^{m_i} + \cdots + h_{i,0}(\underline{T})$ for $i = 1,\ldots,s$ be a sequence of polynomials in $n+1$ variables with coefficients in $\mathbb{Z}$, where $\underline{T} = (T_1,\ldots,T_n)$. Let ϵ be a function from $\{1,\ldots,s\}$ to $\{-1,0,1\}$. Then there exists a boolean combination $\mathcal{B}(\underline{T})$ (i.e. a finite composition of conjunctions, disjunctions and negations) of polynomial equations and inequalities in the variables $\underline{T}$ with coefficients in $\mathbb{Z}$ such that for every real closed field R and for every $\underline{t} \in R^n$, the system*

$$\left\{\begin{array}{lcl} Sign f_1(\underline{t},X) & = & \epsilon(1) \\ & \vdots & \\ Sign f_s(\underline{t},X) & = & \epsilon(s) \end{array}\right.$$

has a solution x in R if and only if $\mathcal{B}(\underline{t})$ holds true in R.

1.

$$f(\underline{X}) := X_1^3 + X_2X_1^2 + X_3X_1 + X_4 = 0 \tag{1}$$

(a) We put $X_1 = X - \dfrac{X_2}{3}$. Then 1 is equivalent to

$$g(X,Y,Z) := X^3 + YX + Z = 0. \tag{2}$$

with

$$\left\{\begin{array}{lcl} X & = & X_1 + \frac{1}{3}X_2; \\ Y & = & -\frac{1}{3}X_2^2 + X_3; \\ Z & = & \frac{2}{27}X_2^3 - \frac{1}{3}X_2X_3 + X_4. \end{array}\right.$$

(b) Put $X = U + V$. Then 2 becomes

$$U^3 + V^3 + (3UV + Y)(U + V) + Z = 0.$$

Then, put $Y = -3UV$, which is equivalent to $U^3V^3 = \dfrac{-Y^3}{27}$. So we have

$$\begin{cases} U^3+V^3 &= -Z; \\ U^3V^3 &= \dfrac{-Y^3}{27}. \end{cases}$$

Thus, by the formulas "coefficients-roots", U^3 and V^3 are the two solutions of the following quadratic equation

$$h(T,Y,Z) := T^2 + ZT - \frac{Y^3}{27} = 0. \tag{3}$$

(c) Write $D := Z^2 + \frac{4}{27}Y^3$, $R[\sqrt{-1}]$ the algebraic closure of R and $j = \dfrac{-1+i\sqrt{3}}{2}$ the classical third rood of 1. Then we have three cases:

- if $D > 0 \Leftrightarrow 27Z^2 + 4Y^3 > 0$, then 3 has two real solutions $U^3 = \dfrac{-Z-\sqrt{D}}{2}$ and $V^3 = \dfrac{-Z+\sqrt{D}}{2}$. Equivalently,

$$U \in \left\{ j^k \sqrt[3]{\frac{-Z-\sqrt{D}}{2}}, k = 0,1,2 \right\};$$
$$V \in \left\{ j^k \sqrt[3]{\frac{-Z+\sqrt{D}}{2}}, k = 0,1,2 \right\}.$$

  Since $X = U + V$ and $Y = -3UV$, there is only one combination which gives a real solution of 2, namely

$$X = \sqrt[3]{\frac{-Z-\sqrt{D}}{2}} + \sqrt[3]{\frac{-Z+\sqrt{D}}{2}}.$$

- if $D = 0 \Leftrightarrow 27Z^2+4Y^3 = 0$, then 3 has one double solution $U^3 = V^3 = \dfrac{-Z}{2}$. Equivalently,

$$U,V \in \left\{ j^k \sqrt[3]{\frac{-Z}{2}}, k = 0,1,2 \right\}.$$

  Since $X = U + V$ and $Y = -3UV$, there are two combinations which give a real solution of 2, namely

$$\begin{cases} X &= -2\sqrt[3]{\dfrac{Z}{2}}\text{(simple root)}; \\ \text{or} & \\ X &= (j+\bar{j})\sqrt[3]{\dfrac{-Z}{2}} \\ &= (\bar{j}+j)\sqrt[3]{\dfrac{-Z}{2}} \\ &= \sqrt[3]{\dfrac{Z}{2}} \end{cases}$$

– if $D < 0 \Leftrightarrow 27Z^2 + 4Y^3 < 0$, then 3 has two complex conjugated solutions $U^3 = \overline{V}^3 = \dfrac{-Z + i\sqrt{-D}}{2}$. Equivalently,

$$U \in \left\{ j^k\rho,\, k = 0,1,2 \right\} \text{ and } V \in \left\{ j^k\overline{\rho},\, k = 0,1,2 \right\}$$

where $\rho := \sqrt[3]{\dfrac{-Z - i\sqrt{D}}{2}}$. Since $X = U + V$ and $Y = -3UV$, there are three combinations which give a real solution of 2, namely

$$\begin{cases} X &= \rho + \overline{\rho}; \\ \text{or} & \\ X &= j\rho + \overline{j\rho} \\ \text{or} & \\ X &= \overline{j}\rho + j\overline{\rho} \end{cases}$$

(d) We just showed that we have the corresponding number of roots for X depending on whether $D(Y,Z) = 27Z^2 + 4Y^3 > 0, = 0$ or < 0. But we know that

$$\begin{cases} X &= X_1 + \frac{1}{3}X_2; \\ Y &= -\frac{1}{3}X_2^2 + X_3; \\ Z &= \frac{2}{27}X_2^3 - \frac{1}{3}X_2X_3 + X_4. \end{cases}$$

So we have the corresponding number of roots depending on the sign of

$$\begin{aligned} \tilde{f}(X_2,X_3,X_4) &:= 27D(-\frac{1}{3}X_2^2 + X_3, \frac{2}{27}X_2^3 - \frac{1}{3}X_2X_3 + X_4) \\ &= 8X_2^6 + 189X_2^2X_3^2 + 108X_3^3X_4 + 27X_3^3 - 487X_2X_3X_4 + 729X_4^2. \end{aligned}$$

2. Consider the polynomials $\in \mathbb{Z}[T,X]$

$$\begin{cases} f_1(T,X) &= TX^2 + (T+1)X + 1 \\ f_2(T,X) &= X^3 - 3T^2X + 2T^3. \end{cases}$$

(a) We get that

- $f_2'(T,X) = 3X^2 - 3T^2 = 3(X-T)(X+T)$;
- $g_1(T,X) = \dfrac{-3T^3 + T^2 + 3T + 1}{T^2}X + \dfrac{2T^4 + T + 1}{T}$;
- $g_2(T,X) = -2T^2X + 2T^3 = -2T^2(X - T)$.

(b) For f_1, we have a discriminant $\Delta_1 = (T+1)^2 - 4T = T^2 + 2T + 1 - 4T = (T-1)^2 \geq 0$. The 2 X-roots of $f_1(T,X)$ are given by the formulas

$$X^{(1),(2)} \in \{\frac{-(T+1) \pm \sqrt{(T-1)^2}}{2T}\} \Leftrightarrow X^{(1),(2)} \in \{-1, \frac{-1}{T}\}.$$

The X-roots of $f_2'(T,X)$ are $\tilde{X}^{(1),(2)} \in \{-T,T\}$.

The X-root of $g_1(T,X)$ is $X^{(0)} = \dfrac{2T^5 + T^2 + T}{3T^3 - T^2 - 3T - 1}$.

The X-root of $g_2(T,X)$ is $\tilde{X}^{(0)} = T$.

(c) As an example, consider $t >> 1$. Denote by $x_1 < x_2 < \ldots < x_5$ the roots computed in the preceding question, $x_0 := -\infty$, $x_6 := +\infty$, and $I_k :=]x_k, x_{k+1}[$ for $k = 0, \ldots, 5$. Then:

(i) Since $t >> 1$, $X^{(0)}(t) = \dfrac{2t^5 + t^2 + t}{-3t^3 + t^2 + 3t + 1}$ behaves like $\dfrac{2}{3}t^2$. So $X^{(0)}(t) > t$. Moreover, $\dfrac{-1}{t} > -1 > -t$. Thus we get that

$$I_0 < x_1 = -t < I_1 < x_2 = -1 < I_2 < x_3 = \frac{-1}{t} < I_3 < x_4 = t < I_4 < x_5 = X^{(0)}(t) < I_5.$$

We get the following matrix 4×11:

$$\begin{array}{cc} x & \begin{array}{ccccccccccc} I_0 & -t & I_1 & -1 & I_2 & \frac{-1}{t} & I_3 & t & I_4 & X^{(0)}(t) & I_5 \end{array} \\ \mathrm{Sign}_R(f_1, f_2', g_1, g_2) = & \left(\begin{array}{ccccccccccc} 1 & 1 & 1 & 0 & -1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & -1 \end{array} \right). \end{array}$$

(ii) The polynomials f_2' and g_2 have a common root t, so it is a double root for f_2. Moreover, with the notation of the lecture, we have

$$x_{i_1} = -t < x_{i_2} = -1 < x_{i_3} = \frac{-1}{t} < x_{i_4} = t.$$

Then we use the fact that $f_2(x_{i_1}) = g_2(x_{i_1}) > 0$, $f_2'(]-\infty, -t[) > 0$ and f_2 has a root in the interval $]-\infty, -t[$ if and only if $\mathrm{Sign}(f_2'(]-\infty, -t[)).\mathrm{Sign}(g_2(x_{i_1})) = 1$, to show that the third root of f_2 is $x^{(3)} \in]-\infty, -t[$.

Now we write

$$y_0 = -\infty < y_1 = x^{(3)} < x_{i_1} = -t < y_2 = x_{i_2} = -1 < y_3 = x_{i_3} = \frac{-1}{t} < y_4 = x_{i_2} = t < y_5 = +\infty.$$

With the notations of the lecture, we have

$$\rho(1) = (0,1), \rho(2) = 2, \rho(3) = 3 \text{ and } \rho(4) = 4.$$

Thus we apply the various criterions (see Bochnak-Coste-Roy) to get that:

- $\mathrm{Sign}(f_2(y_2)) = \mathrm{Sign}(f_2(]y_2, y_3[)) = \mathrm{Sign}(g_1(x_2)) = 1$ and $\mathrm{Sign}(f_2(y_3)) = \mathrm{Sign}(f_2(]y_3, y_4[)) = \mathrm{Sign}(g_1(x_3)) = 1$;
- $\mathrm{Sign}(f_2(y_4)) = 0$ and $\mathrm{Sign}(f_2(]y_4, y_5[)) = \mathrm{Sign}(f_2'(]y_4, y_5[)) = 1$;
- $\mathrm{Sign}(f_2(y_1)) = 0$ and $\mathrm{Sign}(f_2(]y_0, y_1[)) = -\mathrm{Sign}(f_2'(]y_0, y_1[)) = -1$;

$$\begin{array}{cc} x & \begin{array}{ccccccccc} \tilde{I}_0 & x^{(3)} & \tilde{I}_1 & -1 & \tilde{I}_2 & \frac{-1}{t} & \tilde{I}_3 & t & \tilde{I}_4 \end{array} \\ \mathrm{Sign}_R(f_1, f_2) = & \left(\begin{array}{ccccccccc} 1 & 1 & 1 & 0 & -1 & 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \end{array} \right). \end{array}$$

(d) We denote

$$\begin{cases} \tilde{f}_1(T,X) &=& (-3T^3 + T^2 + 3T + 1)X + 2T^5 + T^2 + T; \\ \tilde{f}_2(T,X) &=& -2T^2X + 2T^3 = -2T^2(X-T); \\ \tilde{f}_3(T,X) &=& 3X^2 - 3T^2 = 3(X-T)(X+T); \\ \tilde{f}_4(T,X) &=& TX^2 + (T+1)X + 1. \end{cases}$$

We denote $P(T) := -3T^3 + T^2 + 3T + 1$ and $Q(T) := -2T^6 - 3T^4 - 3T^3 + 3T^2 + 4T + 1$ which are polynomials in $\mathbb{Z}[T]$. Computing the remainders from the corresponding euclidean divisions we obtain

$$\begin{cases} \tilde{f}_1(T,X) &=& P(T)X + 2T^5 + T^2 + T; \\ \tilde{f}_2(T,X) &=& -2T^2X + 2T^3 = -2T^2(X-T); \\ \tilde{f}_3(T,X) &=& 3X^2 - 3T^2 = 3(X-T)(X+T); \\ \tilde{f}'_4(T,X) &=& 2TX + T + 1; \\ \tilde{g}_1(T,X) &=& \dfrac{(2T^5 + T^2 + T)Q(T)}{P(T)^2}; \\ \tilde{g}_2(T,X) &=& T^3 - T^2 - T + 1; \\ \tilde{g}_3(T,X) &=& (T+1)X + T^3 + 1; \\ \tilde{g}_4(T,X) &=& \dfrac{T^3 + T - 2}{4T}. \end{cases}$$

We denote

$$\begin{cases} \tilde{f}^{(1)}_1(T,X) &=& (2T^5 + T^2 + T)Q(T) \text{ (after multiplication of } \tilde{g}_1 \text{ by } P(T)^2); \\ \tilde{f}^{(1)}_2(T,X) &=& T^3 - T^2 - T + 1; \\ \tilde{f}^{(1)}_3(T,X) &=& T^4 + T^2 - 2T \text{ (after multiplication of } \tilde{g}_4 \text{ by } 4T^2); \\ \tilde{f}^{(1)}_4(T,X) &=& P(T)X + 2T^5 + T^2 + T; \\ \tilde{f}^{(1)}_5(T,X) &=& -2T^2X + 2T^3 = -2T^2(X-T); \\ \tilde{f}^{(1)}_6(T,X) &=& 2TX + T + 1; \\ \tilde{f}^{(1)}_7(T,X) &=& (T+1)X + T^3 + 1; \\ \tilde{f}^{(1)}_8(T,X) &=& 3X^2 - 3T^2 = 3(X-T)(X+T); \end{cases}$$

Computing the remainders from the corresponding euclidean divisions (NB : there is no computation with the first polynomials which are of degree 0 in X), we obtain

$$
\left\{
\begin{array}{lcl}
\tilde{f}_1^{(1)}(T,X) & = & (2T^5 + T^2 + T)Q(T); \\
\tilde{f}_2^{(1)}(T,X) & = & T^3 - T^2 - T + 1; \\
\tilde{f}_3^{(1)}(T,X) & = & T^4 + T^2 - 2T; \\
\tilde{f}_4^{(1)}(T,X) & = & P(T)X + 2T^5 + T^2 + T; \\
\tilde{f}_5^{(1)}(T,X) & = & -2T^2X + 2T^3 = -2T^2(X - T); \\
\tilde{f}_6^{(1)}(T,X) & = & 2TX + T + 1; \\
\tilde{f}_7^{(1)}(T,X) & = & (T + 1)X + T^3 + 1; \\
(\tilde{f}_8^{(1)})'(T,X) & = & 6X; \\
\tilde{g}_4^{(1)}(T,X) & = & \dfrac{3(T^5 + T^2 + T)^2 - 3T^2P(T)^2}{P(T)^2}; \\
\tilde{g}_5^{(1)}(T,X) & = & 0 \ \rightarrow \ \text{delete such a trivial equation (see Bochnak-Coste-Roy)}; \\
\tilde{g}_6^{(1)}(T,X) & = & \dfrac{-12T^4 - 3T^2 - 6T - 3}{4T^2}; \\
\tilde{g}_7^{(1)}(T,X) & = & \dfrac{3T^6 - 3T^4 - 4T^3 - 3T^2 + 1}{(T + 1)^2}; \\
\tilde{g}_8^{(1)}(T,X) & = & \dfrac{-3}{2}T^2.
\end{array}
\right.
$$

Once we obtain a polynomial which do not depend on X, it is not used anymore for computations. So from now on we do not write such polynomials. But one may have in mind that they intervene in general for the computations of the corresponding sign matrices.

$$
\left\{
\begin{array}{lcl}
\tilde{f}_1^{(2)}(T,X) & = & P(T)X + 2T^5 + T^2 + T; \\
\tilde{f}_2^{(2)}(T,X) & = & -2T^2X + 2T^3 = -2T^2(X - T); \\
\tilde{f}_3^{(2)}(T,X) & = & 2TX + T + 1; \\
\tilde{f}_4^{(2)}(T,X) & = & (T + 1)X + T^3 + 1; \\
\tilde{f}_5^{(2)}(T,X) & = & 6X;
\end{array}
\right.
$$

Then, computing the remainders from the corresponding euclidean divisions, we obtain

$$
\left\{
\begin{array}{lcl}
\tilde{f}_1^{(2)}(T,X) & = & P(T)X + 2T^5 + T^2 + T; \\
\tilde{f}_2^{(2)}(T,X) & = & -2T^2X + 2T^3 = -2T^2(X - T); \\
\tilde{f}_3^{(2)}(T,X) & = & 2TX + T + 1; \\
\tilde{f}_4^{(2)}(T,X) & = & (T + 1)X + T^3 + 1; \\
(\tilde{f}_5^{(2)})'(T,X) & = & 6; \\
\tilde{g}_1^{(2)}(T,X) & = & \dfrac{-12T^5 - 6T^2 - 6T}{P(T)}; \\
\tilde{g}_2^{(2)}(T,X) & = & 6T; \\
\tilde{g}_3^{(2)}(T,X) & = & \dfrac{-3T - 3}{T}; \\
\tilde{g}_4^{(2)}(T,X) & = & \dfrac{-6T^3 - 6}{T + 1};
\end{array}
\right.
$$

(There is no Euclidean division with $(\tilde{f}_5^{(2)})'$ since it does not depend on X anymore).

It remains

$$\begin{cases} \tilde{f}_1^{(3)}(T,X) &=& P(T)X + 2T^5 + T^2 + T; \\ \tilde{f}_2^{(3)}(T,X) &=& -2T^2X + 2T^3 = -2T^2(X-T); \\ \tilde{f}_3^{(3)}(T,X) &=& 2TX + T + 1; \\ \tilde{f}_4^{(3)}(T,X) &=& (T+1)X + T^3 + 1; \end{cases}$$

which give

$$\begin{cases} \tilde{f}_1^{(3)}(T,X) &=& P(T)X + 2T^5 + T^2 + T; \\ \tilde{f}_2^{(3)}(T,X) &=& -2T^2X + 2T^3 = -2T^2(X-T); \\ \tilde{f}_3^{(3)}(T,X) &=& 2TX + T + 1; \\ \tilde{f}_4^{(3)}(T,X) &=& (T+1)X + T^3 + 1. \end{cases}$$

We need 3 steps more to eliminate completely X Thus we reduced the problem to computing the sign matrix of polynomials $\tilde{f}_j^{(6)}(T)$, $j = 1, \ldots, 23$.

(e) For f_1, as we computed before, the discriminant is $\Delta_1 = (T+1)^2 - 4T = T^2 + 2T + 1 - 4T = (T-1)^2 \geq 0$ and the 2 X-roots are given by the formulas

$$X^{(1),(2)} \in \{\frac{-(T+1) \pm \sqrt{(T-1)^2}}{2T}\} \Leftrightarrow X^{(1),(2)} \in \{-1, \frac{-1}{T}\}.$$

For f_2, the disciminant is $D = 4T^6 + \frac{4}{27}(-27T^6) = 0$. So f_2 has two real roots given by $\tilde{X}_1, \tilde{X}_2 \in \{-2T, T\}$. Now we consider the different cases when evaluating at $t \in R$:

– if $t = 0$, then $f_1(0,X) = X + 1$ and $f_2(0,X) = X^3$ which have sign matrix

$$\begin{array}{cc} x & \begin{array}{ccccc} \tilde{I}_0 & -1 & \tilde{I}_1 & 0 & \tilde{I}_2 \end{array} \\ \mathrm{Sign}_R(f_1,f_2) = & \begin{pmatrix} -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 & 1 \end{pmatrix}. \end{array}$$

– if $t < -1$ then we have $t < -1 < \frac{-1}{t} < -2t$. So $f_1(t,X)$ and $f_2(t,X)$ have sign matrix

$$\begin{array}{cc} x & \begin{array}{ccccccccc} \tilde{I}_0 & t & \tilde{I}_1 & -1 & \tilde{I}_2 & \frac{-1}{t} & \tilde{I}_3 & -2t & \tilde{I}_3 \end{array} \\ \mathrm{Sign}_R(f_1,f_2) = & \begin{pmatrix} -1 & -1 & -1 & 0 & 1 & 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 1 \end{pmatrix}. \end{array}$$

– if $t = -1$ (case where $f_1(-1,X)$ and $f_2(-1,X)$ have as common root -1), then $f_1(0,X) = -X^2 + 1 = (1-X)(1+X)$ and $f_2(0,X) = X^3 - 3X - 2$ have sign matrix

$$\begin{array}{cc} x & \begin{array}{ccccccc} \tilde{I}_0 & -1 & \tilde{I}_1 & 1 & \tilde{I}_2 & 2 & \tilde{I}_3 \end{array} \\ \mathrm{Sign}_R(f_1,f_2) = & \begin{pmatrix} -1 & 0 & 1 & 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 & -1 & 0 & 1 \end{pmatrix}. \end{array}$$

– if $-1 < t < -\frac{1}{\sqrt{2}}$, then $-1 < t < -\frac{-1}{t} < -2t$. So $f_1(t,X)$ and $f_2(t,X)$ have sign matrix

$$\begin{array}{cc} x & \begin{array}{ccccccccc} \tilde{I}_0 & -1 & \tilde{I}_1 & t & \tilde{I}_2 & \frac{-1}{t} & \tilde{I}_3 & -2t & \tilde{I}_3 \end{array} \\ \mathrm{Sign}_R(f_1,f_2) = & \begin{pmatrix} -1 & 0 & -1 & -1 & 1 & 0 & -1 & -1 & -1 \\ -1 & -1 & -1 & 0 & -1 & -1 & -1 & 0 & 1 \end{pmatrix}. \end{array}$$

– if $t = -\dfrac{1}{\sqrt{2}}$, then $f_1(t,X)$ and $f_2(t,X)$ have $\sqrt{2}$ as a common root. So they have matrix sign

$$\begin{array}{cc} x & \begin{array}{ccccccc} \tilde{I}_0 & -1 & \tilde{I}_1 & t & \tilde{I}_2 & \sqrt{2} & \tilde{I}_3 \end{array} \\ \mathrm{Sign}_R(f_1,f_2) = & \begin{pmatrix} -1 & 0 & 1 & 1 & 1 & 0 & -1 \\ -1 & -1 & -1 & 0 & -1 & 0 & 1 \end{pmatrix}. \end{array}$$

– if $-\dfrac{1}{\sqrt{2}} < t < 0$, then $-1 < t < -2t < \dfrac{-1}{t}$. So $f_1(t,X)$ and $f_2(t,X)$ have sign matrix

$$\begin{array}{cc} x & \begin{array}{ccccccccc} \tilde{I}_0 & -1 & \tilde{I}_1 & t & \tilde{I}_2 & 2t & \tilde{I}_3 & \frac{-1}{t} & \tilde{I}_3 \end{array} \\ \mathrm{Sign}_R(f_1,f_2) = & \begin{pmatrix} -1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & -1 \\ -1 & -1 & -1 & 0 & -1 & 0 & 1 & 1 & 1 \end{pmatrix}. \end{array}$$

– if $0 < t < \dfrac{1}{2}$, then $\dfrac{-1}{t} < -1 < -2t < t$. Then we have

$$\begin{array}{cc} x & \begin{array}{ccccccccc} \tilde{I}_0 & \frac{-1}{t} & \tilde{I}_1 & -1 & \tilde{I}_2 & -2t & \tilde{I}_3 & t & \tilde{I}_3 \end{array} \\ \mathrm{Sign}_R(f_1,f_2) = & \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 0 & 1 & 0 & 1 \end{pmatrix}. \end{array}$$

– if $t = \dfrac{1}{2}$, then $f_1(t,X)$ and $f_2(t,X)$ have -1 as a common root. So they have matrix sign

$$\begin{array}{cc} x & \begin{array}{ccccccc} \tilde{I}_0 & -2 & \tilde{I}_1 & -1 & \tilde{I}_2 & \frac{1}{2} & \tilde{I}_3 \end{array} \\ \mathrm{Sign}_R(f_1,f_2) = & \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 & -1 & 0 & 1 \end{pmatrix}. \end{array}$$

– if $\dfrac{1}{2} < t < \dfrac{1}{\sqrt{2}}$, then $\dfrac{-1}{t} < -2t < -1 < t$. Then we have

$$\begin{array}{cc} x & \begin{array}{ccccccccc} \tilde{I}_0 & \frac{-1}{t} & \tilde{I}_1 & -2t & \tilde{I}_2 & -1 & \tilde{I}_3 & t & \tilde{I}_3 \end{array} \\ \mathrm{Sign}_R(f_1,f_2) = & \begin{pmatrix} 1 & 0 & -1 & -1 & -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}. \end{array}$$

– if $t = \dfrac{1}{\sqrt{2}}$ we have $-\sqrt{2}$ common root of f_1, f_2. So we have

$$\begin{array}{cc} x & \begin{array}{ccccccc} \tilde{I}_0 & -\sqrt{2} & \tilde{I}_1 & -1 & \tilde{I}_2 & \frac{1}{\sqrt{2}} & \tilde{I}_3 \end{array} \\ \mathrm{Sign}_R(f_1,f_2) = & \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}. \end{array}$$

– if $\dfrac{1}{\sqrt{2}} < t < 1$, then $-2t < \dfrac{-1}{t} < -1 < t$. Then we have

$$\begin{array}{cc} x & \begin{array}{ccccccccc} \tilde{I}_0 & -2t & \tilde{I}_1 & \frac{-1}{t} & \tilde{I}_2 & -1 & \tilde{I}_3 & t & \tilde{I}_3 \end{array} \\ \mathrm{Sign}_R(f_1,f_2) = & \begin{pmatrix} 1 & 1 & 1 & 0 & -1 & 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}. \end{array}$$

– if $t = 1$ (particular case where $f_1(1,X)$ has only one double root $x_0 = -1$) then $f_1(0,X) = X^2 + 2X + 1 = (X+1)^2$ and $f_2(0,X) = X^3 - 3X + 2$ have sign matrix

$$\begin{array}{cc} x & \begin{array}{ccccccc} \tilde{I}_0 & -2 & \tilde{I}_1 & -1 & \tilde{I}_2 & 1 & \tilde{I}_3 \end{array} \\ \mathrm{Sign}_R(f_1,f_2) = & \left(\begin{array}{ccccccc} 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right). \end{array}$$

– if $1 < t$, then $-2t < -1 < \dfrac{-1}{t} < t$. Then we have

$$\begin{array}{cc} x & \begin{array}{ccccccccc} \tilde{I}_0 & -2t & \tilde{I}_1 & -1 & \tilde{I}_2 & \frac{-1}{t} & \tilde{I}_3 & t & \tilde{I}_3 \end{array} \\ \mathrm{Sign}_R(f_1,f_2) = & \left(\begin{array}{ccccccccc} 1 & 1 & 1 & 0 & -1 & 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \end{array}\right). \end{array}$$

(f) For any real closed field R and any $t \in R$, the semi-algebraic system

$$(I) \quad \left\{ \begin{array}{lcll} f_1(t,X) & = & tX^2 + (t+1)X + 1 & \triangleright_1 \; 0 \\ f_2(t,X) & = & X^3 + 3t^2X + 2t^3 & \triangleright_2 \; 0. \end{array} \right.$$

can be written

$$(I) \quad \left\{ \begin{array}{lcl} \mathrm{Sign}(f_1(t,X)) & = & \epsilon(1) \\ \mathrm{Sign}(f_2(t,X)) & = & \epsilon(2). \end{array} \right.$$

where $\epsilon(1),\epsilon(2) \in \{-1,0,1\}$. Thus, the solutions of the system (I) are the $x \in \mathbb{R}$ for which

$$\left(\begin{array}{c} \epsilon(1) \\ \epsilon(2) \end{array}\right)$$

is a column of any of the sign matrices computed in the preceding question. But the computation of these matrices just rely on finitely many equalities and inequalities for t, i.e. on a semi-algebraic system for T.