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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 7 - Solution

1. (a) For any polynomial $f(\underline{X}) \in R[\underline{X}]$, the corresponding polynomial function is continuous for the interval topology on R^n. So $U(f) := \{\underline{x} \in R^n \mid f(\underline{x}) > 0\} = f^{-1}(]0, +\infty[)$ is open in this topology as the preimage of $]0, +\infty[$ which is open in R.

   (b) Basic open semi-algebraic sets are of the form
   $$\begin{array}{rcl} U(f_1, \dots, f_p) & := & \{\underline{x} \in R^n \mid f_1(\underline{x}) > 0, \dots, f_p(\underline{x}) > 0\} \\ & = & U(f_1) \cap \dots \cap U(f_p). \end{array}$$
   Given any 2 of them, namely $U(f_1, \dots, f_p)$ and $U(g_1, \dots, g_q)$, then we have
   $$\begin{array}{rcl} U(f_1, \dots, f_p) \cap U(g_1, \dots, g_q) & := & \{\underline{x} \in R^n \mid f_1(\underline{x}) > 0, \dots, f_p(\underline{x}) > 0,\} \\ & & \cap\{\underline{x} \in R^n \mid g_1(\underline{x}) > 0, \dots, g_q(\underline{x}) > 0,\} \\ & = & U(f_1) \cap \dots \cap U(f_p) \cap U(g_1) \cap \dots \cap U(g_q) \\ & = & U(f_1, \dots, f_p, g_1, \dots, g_q). \end{array}$$
   Now consider any point $\underline{a} = (a_1, \dots, a_n) \in R^n$. Then the following basic open semi-algebraic set
   $U(f_1, \dots, f_n) = \{\underline{x} \in R^n \mid f_1(\underline{x} = 1 - (x_1 - a_1)^2 > 0, \dots, f_n(\underline{x} = 1 - (x_n - a_n)^2 > 0\}$.
   (It is the open unit cube centered in $\underline{a}$.)

2. Using the normal form for semi-algebraic sets, we only have to prove the statement for basic semi-algebraic sets of the form
   $$Z(f) \cap U(f_1, \dots, f_p).$$
   The set $Z(f)$ is the union of all roots in R of the polynomial $f(\underline{X})$, thus it is as finite set of points.
   For any $i = 1, \dots, p$, write $\alpha_{i,0} = -\infty < \alpha_{i,1} < \dots < \alpha_{i,k_i} < \alpha_{i,k_i+1} = +\infty$ where $\alpha_{i,l}$, $l = 1, \dots, k_i$ are the roots in R of the polynomial $f_i(\underline{X})$. Then, applying the Intermediate Value Theorem, we get that the domain of positivity of $f_i(\underline{X})$, i.e. $U(f_i)$, is a finite union (eventually empty) of intervals $]\alpha_{i,l}, \alpha_{i,l+1}[$, $l = 0, \dots, k_i$.
   To conclude, it suffices to observe that for any $i, j = 1, \dots, p$ and any $l = 0, \dots, k_i$, $m = 0, \dots, k_j$, the intersection $]\alpha_{i,l}, \alpha_{i,l+1}[\cap]\alpha_{j,m}, \alpha_{j,m+1}[$ is again an interval.

3. (a) Consider the horizontal line $\Delta := \{(x,y) \in \mathbb{R}^2 \mid y = 0\}$ which is semi-algebraic. If the infinite zig-zag, call it $\mathcal{Z}$, was semi-algebraic, we would have $\Delta \cap \mathcal{Z} = \{(k,0),\ k \in \mathbb{Z}\}$ semi-algebraic in $\mathbb{R}^2$. Considering the projection on the x-axis of $\Delta \cap \mathcal{Z}$, which is $\mathbb{Z}$, by the geometric version of Tarski-Seidenberg Theorem, we would have $\mathbb{Z}$ semi-algebraic in $\mathbb{R}$. But this contradicts the result of the preceding exercise.

(b) The compact subsets of $\mathbb{R}^2$ are exactly the closed and bounded ones (recall that interval topology is identical to the euclidean topology). So, considering a compact semialgebraic subset K of $\mathbb{R}^2$, we can suppose it included into some closed ball, or equivalently into some closed square

$$T_k := \{(x,y) \in \mathbb{R}^2 \mid k^2 - x^2 \geq 0,\ k^2 - y^2 \geq 0\}$$

for some $k \in \mathbb{N}$. Thus $K \cap \mathcal{Z} = K \cap \mathcal{Z} \cap T_k$.

But $\mathcal{Z} \cap T_k$ is semi-algebraic. Indeed, it is the finite union of segments

$$\{(x,y) \in \mathbb{R}^2 \mid y = (-1)^{l+1}(x - (2l + 1))\} \cap \{(x,y) \in \mathbb{R}^2 \mid 4 - (x - l)^2 \geq 0\}.$$

So $K \cap \mathcal{Z}$ is semi-algebraic.

4. The aim of this exercise is to prove that the real exponential function exp is not semi-algebraic.

(a) Consider some polynomials $p_0(X), \ldots, p_n(X) \in \mathbb{R}[X]$, and an infinite subset $U \subset \mathbb{R}$ such that for all $x \in U$

$$p_n(x)(e^x)^n + p_{n-1}(x)(e^x)^{n-1} + \cdots + p_0(x) = 0.$$

Suppose that the p_i's are not all identically 0, and that n is the biggest exponent of e^x for which p_n is non 0.

(i) If U has no bound, then it has an infinite subsequence, say $(x_k)_{k\in\mathbb{N}}$ tending to $\pm\infty$. For instance, consider the case $x_k \to +\infty$. Write

$$\begin{aligned} f(x) &= p_n(x)(e^x)^n + p_{n-1}(x)(e^x)^{n-1} + \cdots + p_0(x) \\ &= (e^x)^n \left[p_n(x) + \frac{p_{n-1}}{e^x} + \cdots + \frac{p_0}{(e^x)^n} \right]. \end{aligned}$$

But, for any $l = 1, \ldots, n$, we have

$$\lim_{k\to\infty} \frac{p_{n-l}(x_k)}{(e^{x_k})^l} = 0.$$

So we have

$$\lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} (e^{x_k})^n p_n(x_k) = \pm\infty,$$

which contradicts the fact that for any k, $f(x_k) = 0$.

(ii) if U is bounded, since it is infiniteb in $\mathbb{R}$, it has an accumulation point. Now, consider $f(z)$ where z is a complex variable. It is a holomorphic function on the whole complex plane $\mathbb{C}$ as sum and product of e^z, $p_0(z), \ldots, p_n(z)$ which are holomorphic functions on $\mathbb{C}$. Then, applying the Identity Theorem of Complex Analysis, we obtain that $f(z) = 0$ for any $z \in \mathbb{C}$. In particular, restricting to the real variable x, we get that $f(x) = 0$ for any $x \in \mathbb{R}$. Then we can apply the same argument as in the preceding point (i) to get a contradiction.

(b) Suppose that $\Gamma_{\exp}$ is semi-algebraic in $\mathbb{R}^2$. Using the cited Normal Form proposition, we would have that $\Gamma_{\exp}$ is a finite union of basic semi-algebraic sets of the form

$$Z(g) \cap U(g_1, \dots, g_p).$$

for $g, g_1, \dots, g_p \in \mathbb{R}[X,Y]$.

Now, we proved in the preceding question that $Z(g)$ must be finite for any g. So $\Gamma_{\exp}$ would be piecewise a finite union of sets $U(g_1, \dots, g_p)$. But in exercise 1, we proved that for any g, $U(g)$ is open in $\mathbb{R}^2$, so $U(g_1, \dots, g_p)$ is open in $\mathbb{R}^2$. This would mean that $\Gamma_{\exp}$ contains an open square of $\mathbb{R}^2$, and thus has non empty interior. This is false, since it is a graph of a continuous one variable function, i.e. it is a closed set with empty interior.