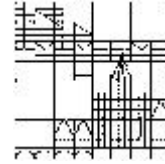


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## Übungen zur Vorlesung Reelle algebraische Geometrie

### Blatt 6

*These exercises will be collected Tuesday 01 December in the mailbox number 15 of the Mathematics department.*

1. Let  $(K, \leq)$  be an ordered field which verifies the **intermediate value property** (**Zwischenwerteigenschaft**):

$$\forall a < b \in K, f(a) < 0 < f(b) \Rightarrow \exists c \in ]a, b[, f(c) = 0.$$

(a) Show that any positive element of  $K$  is a square.

(b) Show that any polynomial of  $K[X]$  with odd degree has a root in  $K$ . Then conclude that  $K$  is **real closed**.

**Definition 0.1** Let  $G := (G, \cdot)$  be a group. A total ordering  $\leq$  of the set  $G$  is said to be a **group ordering** of  $G$  if for any  $g_1, g_2, h \in G$ , we have

$$g_1 \geq g_2 \Rightarrow g_1 \cdot h \geq g_2 \cdot h \text{ and } h \cdot g_1 \geq h \cdot g_2.$$

Then one says that  $G$  is an **ordered group**.

2. Let  $(R, \leq)$  be a real closed field. Consider

$$Pos(R) := \{x \in R \mid 0 < x\}.$$

(a) Show that  $(Pos(R), \cdot)$  is a totally ordered abelian subgroup of  $(R^*, \cdot)$ .

(b) Show that  $(Pos(R), \cdot)$  is **divisible**, i.e. for any  $a \in Pos(R)$ , we have  $\sqrt[n]{a} \in Pos(R)$  for any  $n \in \mathbb{N}$ .

3. We consider the **Motzkin polynomial**

$$m(X, Y) = 1 - 3X^2Y^2 + X^2Y^4 + X^4Y^2.$$

(a) Show that  $m(X, Y) \geq 0$  on  $\mathbb{R}^2$ .

(Hint: use the following inequality between the arithmetical mean and the geo-

metrical mean:  $\forall a, b, c \geq 0, \frac{a+b+c}{3} \geq \sqrt[3]{abc}$ .)

(b) Denote  $\underline{X} = (X_1, \dots, X_n)$  for some  $n \in \mathbb{N}^*$  and consider  $f(\underline{X}) \in \mathbb{R}[\underline{X}]$ . Show that, if  $f = f_1^2 + \dots + f_k^2$  for some  $f_i(\underline{X}) \in \mathbb{R}[\underline{X}]$  with  $f_1 \neq 0$ , then  $f \neq 0$  and  $\deg(f) = 2 \max\{\deg(f_i) \mid i = 1, \dots, k\}$ .

(c) Suppose that the Motzkin polynomial  $m = f_1^2 + \dots + f_k^2$  for some  $f_i(X, Y) \in \mathbb{R}[X, Y]$ . Deduce that for any  $i = 1, \dots, k$ ,  $f_i(X, Y)$  is a polynomial of degree at most 3 and that it can contain none of the monomials  $X^3, Y^3, X^2, Y^2, X$  and  $Y$ .

(d) Conclude by contradiction that  $m$  cannot be a sum of squares in  $\mathbb{R}[X, Y]$ .  
(Hint: write  $f_i(X, Y) = a_i + b_iXY + c_iX^2Y + d_iXY^2$  for any  $i = 1, \dots, k$ .)

4. Denote  $\underline{X} = (X_1, \dots, X_n)$  for some fixed  $n \in \mathbb{N}^*$  and let  $d \in \mathbb{N}$ . Consider some non zero polynomial  $f(\underline{X}) \in \mathbb{R}[\underline{X}]$  of total degree less than or equal to  $d$ .

(a) Show that

$$\bar{f}(X_0, X_1, \dots, X_n) = X_0^d f\left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right)$$

is a homogenous polynomial (or form) of degree  $d$  in the variables  $X_0, \dots, X_n$ .

**Definition 0.2** We call  $\bar{f}(X_0, X_1, \dots, X_n)$  the **homogenization** of  $f(\underline{X})$ .

(b) Denote by  $V_{d,n}$  the  $\mathbb{R}$ -vector space of all polynomials of degree  $\leq d$  in  $\mathbb{R}[X_1, \dots, X_n]$ , and by  $F_{d,n+1}$  the  $\mathbb{R}$ -vector space of all homogenous polynomials of degree  $d$  in  $\mathbb{R}[X_0, X_1, \dots, X_n]$ . Show that the homogenization map

$$f(X_1, \dots, X_n) \mapsto \bar{f}(X_0, X_1, \dots, X_n)$$

is a vector space isomorphism between  $V_{d,n}$  and  $F_{d,n+1}$ .

(c) From now on, we suppose that  $d$  is even. Show that  $f \geq 0$  on  $\mathbb{R}^n$  if and only if  $\bar{f} \geq 0$  on  $\mathbb{R}^{n+1}$ .

(d) Show that  $f$  is a sum of squares of polynomials if and only if  $\bar{f}$  is a sum of squares of homogenous polynomials of degree  $d/2$ .

(e) Denote  $\tilde{P}_{d,n}$  the subset of polynomials in  $\mathbb{R}[\underline{X}]$  of degree  $\leq d$  and which are **positive semidefinite** (i.e  $f \geq 0$ ), and  $\tilde{\Sigma}_{d,n}$  its subset of polynomials which are sums of squares. Give a version of **Hilbert's Theorem on Positive Semidefinite forms** in this context.

(f) Show that the homogenous Motzkin polynomial

$$\bar{m}(X, Y, Z) = Z^6 + X^4Y^2 + X^2Y^4 - 3X^2Y^2Z^2$$

is positive semi-definite on  $\mathbb{R}^3$  but is not a sum of squares.

(Hint: use exercise 3.)