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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 6

These exercises will be collected Tuesday 01 December in the mailbox number 15 of the Mathematics department.

1. Let $(K, \leq)$ be an ordered field which verifies the **intermediate value property** (**Zwischenwerteigenschaft**):
$$\forall a < b \in K,\ f(a) < 0 < f(b) \ \Rightarrow\ \exists c \in]a,b[,\ f(c) = 0.$$
(a) Show that any positive element of K is a square.

(b) Show that any polynomial of $K[X]$ with odd degree has a root in K. Then conclude that K is **real closed**.

Definition 0.1 *Let $G := (G,.)$ be a group. A total ordering $\leq$ of the set G is said to be a* ***group ordering*** *of G if for any $g_1,g_2,h \in G$, we have*
$$g_1 \geq g_2 \ \Rightarrow\ g_1.h \geq g_2.h \text{ and } h.g_1 \geq h.g_2.$$
Then one says that G is an ***ordered group****.*

2. Let $(R, \leq)$ be a real closed field. Consider
$$Pos(R) := \{x \in R \mid 0 < x\}.$$
(a) Show that $(Pos(R),.)$ is a totally ordered abelian subgroup of $(\mathbb{R}^*,.)$.

(b) Show that $(Pos(R),.)$ is **divisible**, i.e. for any $a \in Pos(R)$, we have $\sqrt[n]{a} \in Pos(R)$ for any $n \in \mathbb{N}$.

3. We consider the **Motzkin polynomial**
$$m(X,Y) = 1 - 3X^2Y^2 + X^2Y^4 + X^4Y^2.$$
(a) Show that $m(X,Y) \geq 0$ on $\mathbb{R}^2$.
(*Hint: use the following inequality between the arithmetical mean and the geometrical mean:* $\forall a,b,c \geq 0,\ \dfrac{a+b+c}{3} \geq \sqrt[3]{abc}$.)

(b) Denote $\underline{X} = (X_1, \ldots, X_n)$ for some $n \in \mathbb{N}^*$ and consider $f(\underline{X}) \in \mathbb{R}[\underline{X}]$. Show that, if $f = f_1^2 + \cdots + f_k^2$ for some $f_i(\underline{X}) \in \mathbb{R}[\underline{X}]$ with $f_1 \neq 0$, then $f \neq 0$ and $\deg(f) = 2\max\{\deg(f_i) \mid i = 1, \ldots, k\}$.

(c) Suppose that the Motzkin polynomial $m = f_1^2 + \cdots f_k^2$ for some $f_i(X,Y) \in \mathbb{R}[X,Y]$. Deduce that for any $i = 1, \ldots, k$, $f_i(X,Y)$ is a polynomial of degree at most 3 and that it can contain none of the monomials X^3, Y^3, X^2, Y^2, X and Y.

(d) Conclude by contradiction that m cannot be a sum of squares in $\mathbb{R}[X,Y]$.
(*Hint: write* $f_i(X,Y) = a_i + b_i XY + c_i X^2 Y + d_i XY^2$ *for any* $i = 1, \ldots, k$.)

4. Denote $\underline{X} = (X_1, \ldots, X_n)$ for some fixed $n \in \mathbb{N}^*$ and let $d \in \mathbb{N}$. Consider some non zero polynomial $f(\underline{X}) \in \mathbb{R}[\underline{X}]$ of total degree less than or equal to d.

(a) Show that

$$\overline{f}(X_0, X_1, \ldots, X_n) = X_0^d f\left(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}\right)$$

is a homogenous polynomial (or form) of degree d in the variables $X_0, \ldots, X_n$.

Definition 0.2 *We call* $\overline{f}(X_0, X_1, \ldots, X_n)$ *the* ***homogenization*** *of* $f(\underline{X})$.

(b) Denote by $V_{d,n}$ the $\mathbb{R}$-vector space of all polynomials of degree $\leq d$ in $\mathbb{R}[X_1, \ldots, X_n]$, and by $F_{d,n+1}$ the $\mathbb{R}$-vector space of all homogenous polynomials of degree d in $\mathbb{R}[X_0, X_1, \ldots, X_n]$. Show that the homogenization map

$$f(X_1, \ldots, X_n) \mapsto \overline{f}(X_0, X_1, \ldots, X_n)$$

is a vector space isomorphism between $V_{d,n}$ and $F_{d,n+1}$.

(c) From now on, we suppose that d is even. Show that $f \geq 0$ on $\mathbb{R}^n$ if and only if $\overline{f} \geq 0$ on $\mathbb{R}^{n+1}$.

(d) Show that f is a sum of squares of polynomials if and only if $\overline{f}$ is a sum of squares of homogenous polynomials of degree $d/2$.

(e) Denote $\tilde{P}_{d,n}$ the subset of polynomials in $\mathbb{R}[\underline{X}]$ of degree $\leq d$ and which are **positive semidefinite** (i.e $f \geq 0$), and $\tilde{\sum}_{d,n}$ its subset of polynomials which are sums of squares. Give a version of **Hilbert's Theorem on Positive Semidefinite forms** in this context.

(f) Show that the homogenous Motzkin polynomial

$$\overline{m}(X,Y,Z) = Z^6 + X^4Y^2 + X^2Y^4 - 3X^2Y^2Z^2$$

is positive semi-definite on $\mathbb{R}^3$ but is not a sum of squares.
(*Hint: use exercise* 3.)