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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 5

These exercises will be collected Tuesday 24 November in the mailbox number 15 of the Mathematics department.

1. Let $(K, \leq)$ be an ordered field.

(a) Show that the system of all sets

$$]a,b[:= \{x \in K \mid a < x < b\}, \ \ a,b \in K$$

called intervals, together with the empty set is a cover (Überdeckung) of K and is closed under finite intersection (endliche Durchschnitt).

Therefore, this system is the base of a topology on K, which is called the **interval topology**. From now on, we consider K endowed with this interval topology.

(b) Show that the following mappings are continuous (stetig):

(i) the field operations as mappings from $K \times K$ endowed with the product topology, to K;

(ii) the multiplicative inversion from K^* endowed with the induced topology (Spurtopologie), to K.

(c) Show that, if K is Dedekind complete (therefore $K \cong \mathbb{R}$: see Übungsaufgaben Blatt 2), then it is a connected (zusammenhängend) topological space.

(d) Let K be **real closed** (**reell abgeschlossen**). Define on K the euclidean norm

$$\begin{array}{cccc} \| \ \| : & K^n & \to & K \\ & (x_1, \ldots, x_n) & \to & \sqrt{x_1^2 + \cdots + x_n^2}. \end{array}$$

Show that the **euclidean topology** (i.e. the one induced by this norm) on K^n coincides with the product topology on K^n induced by the interval topology.

2. Let R be a real closed field and $f(x) \in R[x]$ be some polynomial with positive degree. Let $c \in R$ be a root of $f(x)$.

   Show that there exists $\delta > 0$ in R such that for any x with $|x - c| < \delta$, we have
   $$Sign(f(x)f'(x)) = Sign(x - c).$$
   (*Hint: reduce to the case* $c = 0$ *by a change of variable* $X = x - c$)

3. Let R be a real closed field and let $f(x) = x^3 + 6x^2 - 16$ in $R[x]$.

   (a) Compute the **Sturm sequence** (**Kette**) of $f(x)$.

   (b) Show that f(x) has 3 roots in R.

   (c) Denote them by $\alpha_1 \leq \alpha_2 \leq \alpha_3$. Show that for all i, $\alpha_i \in [-7,2]$, and that $\alpha_1 \in [-6, -5]$, $\alpha_2 = -2$ and that $\alpha_3 \in [1,2]$.

   Note: the computations have been done and thus the results are valid for *any* real closed field R.

4. Show that $\mathbb{Q}(\sqrt{2})$ has exactly two orderings that extends the one of $\mathbb{Q}$.
   (*Hint: consider the embedding of* $\mathbb{Q}$ *in* $\mathbb{R}$ *and use Corollary 6 of the Lecture of 12/11/09*).

5. Consider the ring of **real formal power series**
   $$\mathbb{R}[[X]] := \left\{ \sum_{i=0}^{\infty} a_i X^i \mid a_i \in \mathbb{R} \right\}$$
   and the field of **real Laurent series**
   $$\mathcal{K} := \left\{ \sum_{i=m}^{\infty} a_i X^i \mid m \in \mathbb{Z}, a_i \in \mathbb{R} \right\}.$$
   (see Übungsaufgaben Blatt 3).

   Show that $\mathcal{K}$ admits exactly two orderings extending the one on $\mathbb{R}$.
   (*Hint: show that power series* $1 + \sum_{i=1}^{\infty} a_i X^i$ *are squares*).