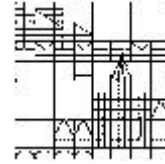


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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 5

These exercises will be collected Tuesday 24 November in the mailbox number 15 of the Mathematics department.

1. Let (K, \leq) be an ordered field.

(a) Show that the system of all sets

$$]a,b[:= \{x \in K \mid a < x < b\}, \quad a, b \in K$$

called intervals, together with the empty set is a cover (Überdeckung) of K and is closed under finite intersection (endliche Durchschnitt).

Therefore, this system is the base of a topology on K , which is called the **interval topology**. From now on, we consider K endowed with this interval topology.

(b) Show that the following mappings are continuous (stetig):

(i) the field operations as mappings from $K \times K$ endowed with the product topology, to K ;

(ii) the multiplicative inversion from K^* endowed with the induced topology (Spurtopologie), to K .

(c) Show that, if K is Dedekind complete (therefore $K \cong \mathbb{R}$: see Übungsaufgaben Blatt 2), then it is a connected (zusammenhängend) topological space.

(d) Let K be **real closed (reell abgeschlossen)**. Define on K the euclidean norm

$$\begin{aligned} \|\cdot\| : \quad K^n &\rightarrow K \\ (x_1, \dots, x_n) &\rightarrow \sqrt{x_1^2 + \dots + x_n^2}. \end{aligned}$$

Show that the **euclidean topology** (i.e. the one induced by this norm) on K^n coincides with the product topology on K^n induced by the interval topology.

2. Let R be a real closed field and $f(x) \in R[x]$ be some polynomial with positive degree. Let $c \in R$ be a root of $f(x)$.

Show that there exists $\delta > 0$ in R such that for any x with $|x - c| < \delta$, we have

$$\text{Sign}(f(x)f'(x)) = \text{Sign}(x - c).$$

(Hint: reduce to the case $c = 0$ by a change of variable $X = x - c$)

3. Let R be a real closed field and let $f(x) = x^3 + 6x^2 - 16$ in $R[x]$.

(a) Compute the **Sturm sequence (Kette)** of $f(x)$.

(b) Show that $f(x)$ has 3 roots in R .

(c) Denote them by $\alpha_1 \leq \alpha_2 \leq \alpha_3$. Show that for all i , $\alpha_i \in [-7, 2]$, and that $\alpha_1 \in [-6, -5]$, $\alpha_2 = -2$ and that $\alpha_3 \in [1, 2]$.

Note: the computations have been done and thus the results are valid for *any* real closed field R .

4. Show that $\mathbb{Q}(\sqrt{2})$ has exactly two orderings that extends the one of \mathbb{Q} .
(Hint: consider the embedding of \mathbb{Q} in \mathbb{R} and use Corollary 6 of the Lecture of 12/11/09).

5. Consider the ring of **real formal power series**

$$\mathbb{R}[[X]] := \left\{ \sum_{i=0}^{\infty} a_i X^i \mid a_i \in \mathbb{R} \right\}$$

and the field of **real Laurent series**

$$\mathcal{K} := \left\{ \sum_{i=m}^{\infty} a_i X^i \mid m \in \mathbb{Z}, a_i \in \mathbb{R} \right\}.$$

(see Übungsaufgaben Blatt 3).

Show that \mathcal{K} admits exactly two orderings extending the one on \mathbb{R} .

(Hint: show that power series $1 + \sum_{i=1}^{\infty} a_i X^i$ are squares).