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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 5 - Solution

1. (a) The intervals cover K: for any $x \in K$, $x \in]x-1, x+1[$.
For finite intersection of intervals, it suffices to consider 2 of them. Verify the case if one of them is the empty set. If not, denote them by $]a,b[$ and $]c,d[$ with $a < b$ and $c < d$ in K and for instance $b - a \geq d - c$, and consider the 4 different cases and compute the intersection (make a picture).

   (b) (i) Consider a point $(a,b) \in K \times K$, we use the definition of continuity at this point. Take any $\epsilon > 0$ in K. Then, for any $(x,y) \in]a - \epsilon/2, a + \epsilon/2[\times]b - \epsilon/2, b + \epsilon/2[$, we have $x + y \in]a + b - \epsilon, a + b + \epsilon[$. For multiplication, consider $0 < \alpha < \min\left\{\sqrt{\frac{\epsilon}{2}}, \frac{\epsilon}{4|b|}\right\}$ and $0 < \beta < \min\left\{\sqrt{\frac{\epsilon}{2}}, \frac{\epsilon}{4|a|}\right\}$. Then for any $(x,y) \in]a-\alpha, a+\alpha[\times]b-\beta, b+\beta[$, we have $x.y \in]a.b - \epsilon, a.b + \epsilon[$ (for the computations, use inequalities with absolute values so that you don't need to consider the different cases).

   (ii) Consider $a \in K^*$ and any $\epsilon > 0$ in K. We look for some $\alpha > 0$ such that, whenever $x \in]|a| - \alpha, |a| + \alpha[$, we have $\frac{1}{x} \in]\frac{1}{|a|} - \epsilon, \frac{1}{|a|}\epsilon[$.
This implies that

$$0 < \alpha < \frac{\epsilon|a|^2}{1 + \epsilon|a|}.$$

   Then it remains to show that this condition is sufficient (note that $\frac{\epsilon|a|^2}{1+\epsilon|a|} < \frac{\epsilon|a|^2}{1-\epsilon|a|}$ since $0 < 1 - \epsilon|a| < 1 + \epsilon|a|$). We suppposed without loss of generality that $\epsilon < \frac{1}{|a|}$.

   (c) We know that the connected subsets of $\mathbb{R}$ are exactly the intervals. Then so it is by isomorphism for K.

Now take any $a \in K$ and consider its connected component C_a. As a connected subset of K, C_a is a non empty (it contains a) interval. Moreover, since any interval in K is connected, then any interval $]a - x, a + x[$ for a positive x is included in C_a since it contains a. Then make x tends to ∞.

(d) It suffices to show that the base for the product topology, namely the hypercubes

$$\prod_{i=1}^{n}]a_i, b_i[\text{ for any } a_i, b_i \in K,$$

is equivalent to the base for the euclidean topology, namely the open balls

$$B((a_1, \dots, a_n), r) := \{(x_1, \dots, x_n) \in K^n \mid \sqrt{(x_1 - a_1)^2 + \dots + (x_n - a_n)^2} < r\} \text{ for any } a_i, r \in K \text{ with } r > 0.$$

Thus, one has to show that, for any such hypercube, there exist a ball contained in it (take $\left(\frac{a_1 + b_1}{2}, \dots, \frac{a_1 + b_1}{2}\right)$ as a center and $\min_i\{\frac{b_i - a_i}{2}\}$ as a radius) and a ball containing it (take the same center and $\max_i\{\frac{b_i - a_i}{2}\}$ as a radius).

2. By the change of variable $X = x - c$, we reduce to the case of a polynomial $F(x) = a_0 X^n + \dots + a_{n-m} X^m$ which has 0 as a root with multiplicity m. We want to show that there exists $\delta > 0$ in R such that for any X with $|X| < \delta$, $Sign(F(X)F'(x)) = Sign(X)$.
We rewrite $F(X) = X^m G(X)$ with $G(X) = a_0 X^{n-m} + \dots + a_{n-m}$ and $G(0) = a_{n-m} \neq 0$. Then we have $XF'(X) = mX^m G(X) + X^m X G'(X) =$ and $G'(X) = (n - m) a_0 X^{n-m-1} + \dots + a_{n-m+1}$. Then we have
$$\frac{XF'(X)}{F(X)} = m + X\frac{G'(X)}{G(X)}.$$
But the second term $X\frac{G'(X)}{G(X)}$ has value $0\frac{G'(0)}{G(0)} = 0\frac{a_{n-m+1}}{a_{n-m}} = 0$ when $X = 0$. By continuity of $X\frac{G'(X)}{G(X)}$, there exists $\delta > 0$ such that for any $|X| < \delta$, we have $\left|X\frac{G'(X)}{G(X)}\right| < m$. Then for any such X, we have $\frac{XF'(X)}{F(X)} > 0$.

3. Consider $f(x) = x^3 + 6x^2 - 16$ in $R[x]$.

(a) The Sturm sequence of $f(x)$ is $S_f(x) = (f_0(x), \dots, f_3(x))$ with:
$$\begin{cases} f_0(x) &= f(x); \\ f_1(x) &= 3x^2 + 12x; \\ f_2(x) &= 8x + 16; \\ f_3(x) &= 12. \end{cases}$$
(b) We have
$$\begin{aligned} V_f(-\infty) &= Var((-1)^3, (-1)^2 3, (-1)^1 8, 12) \\ &= Var(-1, 3, -18, 12) \\ &= 3 \\ V_f(+\infty) &= Var(1, 3, 18, 12) \\ &= 0. \end{aligned}$$

So the number of roots of $f(x)$ in R is $V_f(-\infty) - V_f(+\infty) = 3$.

(c) We compute $S_f(-7) = (-65, 63, -40, 12)$ which has 3 sign changes, and $S_f(2) = (16, 36, 32, 12)$ which has no sign change. Then there are $3 - 0 = 3$ roots between -7 and 2.
We compute $S_f(-6) = (-16, 36, -32, 12)$ which has 3 sign changes, and $S_f(-5) = (9, 15, -24, 12)$ which has 2 sign changes. So there is $3 - 2 = 1$ root, say α_1 between -6 and -5.
We compute $f(-2) = 0$, so $\alpha_2 = -2$.
We compute $S_f(1) = (-9, 15, 24, 12)$ which has 1 sign change. Since $S_f(2)$ has no sign change, the third root α_3 is between 1 and 2.

4. We consider $\mathbb{Q}$ embedded in $\mathbb{R}$ by the inclusion map, say $\phi : \mathbb{Q} \to \mathbb{R}$. Then we consider the algebraic extension $\mathbb{Q}(\sqrt{2})$ of $\mathbb{Q}$, which is a quadratic extension: the minimum polynomial is $f(x) = x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2})$. Then by Corollary 6 of the Lecture, the number of embedding extensions $\psi : \mathbb{Q}(\sqrt{2}) \to \mathbb{R}$ is equal to the number of extensions Q of the ordering $P = \mathbb{Q}_{\geq 0}$. Here we have only two possibilities:
• either $\sqrt{2} \in Q \Leftrightarrow \psi(\sqrt{2}) = \sqrt{2} > 0$ in $\mathbb{R}$ (in this case, ψ is the inclusion as ϕ);
• or $-\sqrt{2} \in Q \Leftrightarrow \psi(\sqrt{2}) = -\sqrt{2} < 0$ in $\mathbb{R}$ (in this case, ψ is order reversing for $-\sqrt{2}$: it looks like conjugation for complex numbers).

5. We consider a series $1 + \sum_{i=1}^{\infty} a_i X^i$. We show that $1 + \sum_{i=1}^{\infty} a_i X^i = (1 + \sum_{i=1}^{\infty} b_i X^i)^2$ for some $b_i \in \mathbb{R}$. Indeed,

$$(1+\sum_{i=1}^{\infty} b_i X^i)^2 = 1+2b_1X+(2b_2+b_1^2)X^2+2(b_3+b_1b_2)X^3+(2b_4+2b_1b_3+b_2^2)X^4+\cdots,$$

and so, by induction, one proves that for any $n \in \mathbb{N}^*$, $2b_n = a_n + P_n(a_{n-1}, \ldots, a_1)$ for some quadratic polynomial P_n in $\mathbb{R}[X]$.
As an example, we compute $b_1 = a_1/2$, $b_2 = (a_2 - a_1^2/4)/2$, $b_3 = a_3/2 - a_1(a_2 - a_1^2/4)/4$.
As a consequence, for any ordering on $\mathcal{K}$ extending the one on the reals, we have $c_0 + c_1X + c_2X^2 \cdots = c_0(1 + \sum_{i=1}^{\infty} a_i X^i) > 0$ if and only if $c_0 > 0$. It implies that X is infinitesimal compared to the reals. Then the two orderings extending the one on $\mathbb{R}$ are given by either $\mathbb{R}_{>0} > X > 0$ or $\mathbb{R}_{<0} < X < 0$.
(One can verify this looking at an arbitrary non zero Laurent series

$$c(X) = c_{-m}X^{-m} + c_{-m+1}X^{-m+1} + \cdots.$$

Factorizing by $c_{-m}X^{-m}$, we rewrite it

$$\begin{aligned} c(X) &= c_{-m}X^{-m}(1 + a_1X + a_2X^2 + \cdots) \text{ with } a_i := c_{-m+i}/c_{-m} \\ &= c_{-m}X^{-m}(1 + \sum_{i=1}^{\infty} b_i X^i)^2.) \end{aligned}$$