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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 4 - Solution.

1. By induction on n, it suffices to consider a field K endowed with an order P, and to show that it extends to an order on $K(x)$.

   (a) The refered result is:
   Let L/K be a field extension and P an order on K. Consider the set
   $$T_L(P) := \left\{ \sum_{i=1}^{n} a_i y_i^2 \mid n \in \mathbb{N}, a_i \in P, y_i \in L \right\}.$$
   Then P admits an extension to an order on L if $-1 \notin T_L(P)$.
   We reason by absurd: suppose that $-1 \in T_{K(x)}(P)$, then we have $p_0^2 + a_1 p_1^2 + \cdots + a_n p_n^2 = 0$ for some $a_i \in P \subset K$ and $p_i \in K[x]$. The leading coefficient of the left term of this equality is of the form $b_0^2 + \sum a_i b_i^2 = 0$ with $b_i \in K$, which would mean that $-1 \in P \to$ contradiction.

   (b) For every $f(x), g(x) \in K[x]$ with $f(x) = d_m x^m + \cdots + d_k x^k$ ($m \geq k$) and $g(x) \neq 0$, define
   $$\begin{aligned} f(x) \geq 0 \quad &\Leftrightarrow \quad d_k \geq 0; \\ \frac{f(x)}{g(x)} \geq 0 \quad &\Leftrightarrow \quad f(x)g(x) \geq 0 \end{aligned}$$
   and show that this is an order on $K(x)$ extending the one on K, by showing that the set of positive elements is a positive cone containing P.
   Note that x is positive infinitesimal with respect to K, i.e. $0 < x < a$ for all $a \in K$. Thus the order on $K(x_1, \ldots, x_n)$ that we get is such that x_{i+1} is infinitesimal with respect to $K(x_1, \ldots, x_i)$ for any $i = 1, \ldots, n-1$.

2. Consider $x \in R$ with $|x| \geq D$. We write
   $$f(x) = dx^m \left(1 + \frac{d_{m-1}}{d} x^{-1} + \cdots + \frac{d_0}{d} x^{-m}\right).$$
   Consider
   $$\left| \frac{d_{m-1}}{d} x^{-1} + \cdots + \frac{d_0}{d} x^{-m} \right|.$$

Since $|x| \geq D \geq 1$, we have $1 \geq D^{-1} \geq |x^{-1}| \geq |x^{-2}| \geq \cdots \geq |x^{-m}|$. Moreover applying the triangular inequality, we get that:

$$\left|\frac{d_{m-1}}{d}x^{-1} + \cdots + \frac{d_0}{d}x^{-m}\right| \leq \left(\left|\frac{d_{m-1}}{d}\right| + \cdots + \left|\frac{d_0}{d}\right|\right) D^{-1}.$$

Since by definition

$$D := 1 + \sum_{i=m-1}^{0} \left|\frac{d_i}{d}\right| > \sum_{i=m-1}^{0} \left|\frac{d_i}{d}\right|.$$

we have

$$\left|\frac{d_{m-1}}{d}x^{-1} + \cdots + \frac{d_0}{d}x^{-m}\right| < 1$$

We deduce that

$$\left|1 + \frac{d_{m-1}}{d}x^{-1} + \cdots + \frac{d_0}{d}x^{-m}\right| > 0$$

and so

$$|f(x)| = |dx^m| \left|1 + \frac{d_{m-1}}{d}x^{-1} + \cdots + \frac{d_0}{d}x^{-m}\right| > 0.$$

3. Let $f(x) = x^m + d_{m-1}x^{m-1} + \cdots + d_0 \in R[x]$ with roots $a_1, \ldots, a_m \in R$.

   If $a_i \geq 0$, for all $i = 1, \ldots, m$, denote n the number of positive roots. Then $n \leq m$. If $n = m$, then by Descartes lemma, there are at least m sign changes in the sequence $(1, d_{m-1}, \ldots, d_0)$ which is only possible if for any i, there is a sign change between 1 and d_{m-1}, which implies that $d_{m-1} < 0 \Leftrightarrow (-1)d_{m-1} > 0$, and between d_{i+1} and d_i, which implies by a straightforward induction that $(-1)^{m-i}d_i \geq 0$ for all i. If $n < m$, it means that 0 is a root with multiplicity $m - n$. Equivalently, we can factor x^{m-n} in $f(x)$. Then we obtain a polynomial of degree n with n positive roots, as in the preceding case.

   If $(-1)^{m-i}d_i \geq 0$ for all $i = 0, \ldots, m-1$, suppose that there exists a negative root $a < 0$. Then $\beta := -a > 0$. We have

$$\begin{aligned} f(a) &= 0 \\ &= f(-\beta) \\ &= (-1)^m\beta^m + (-1)^{m-1}d_{m-1}\beta^{m-1} + \cdots + (-1)d_1\beta + d_0 \\ &= (-1)^m \left[\beta^m + (-1)d_{m-1}\beta^{m-1} + \cdots + (-1)^{m-1}d_1\beta + (-1)^m d_0\right]. \end{aligned}$$

   But a sum of non negative terms is zero if and only if each term is zero $\rightarrow$ contradiction.

4. Consider $f(x) = dx^m + d_{m-1}x^{m-1} + \cdots + d_0 \in R[x]$. We write its decomposition in irreducible factors as

$$f(x) = d \prod (x - a_i)^{k_i} \prod [(x - b_j)^2 + c_j^2]^{l_j}.$$

   We show that $(a) \Rightarrow (b)$. Suppose that $f \geq 0$ on R and that there exists a factor $(x - a_i)$ with multiplicity k_i odd (ungerade), say $i = 1$ for instance. Consider

$$\frac{f(x)}{(x - a_1)^{k_1}} = d \prod_{i \geq 2} (x - a_i)^{k_i} \prod [(x - b_j)^2 + c_j^2]^{l_j}.$$

Since $(x-a_1)^{k_1}$ has a (unique) sign change at a_1 and $f(x) \geq 0$, we should have a sign change at a_1 in the right term of this equality, which is a polynomial. By the Intermediate Value Theorem, this polynomial would have a_1 as a root. Equivalently $(x-a_1)$ would be a factor of it, which contradicts the fact that k_1 is the multiplicity of a_1 for f. Thus all the k_i's are even, so $\prod(x-a_i)^{k_i} \prod[(x-b_j)^2+c_j^2]^{l_j} \geq 0$ and consequently $d > 0$.

We show that $(b) \Rightarrow (c)$. We suppose that $k_i = 2m_i$ for all i. Then we write

$$f(x) = \left[\sqrt{d} \prod_{i \geq 2} (x-a_i)^{m_i}\right]^2 \prod[(x-b_j)^2 + c_j^2]^{l_j}.$$

Now use the fact that for any a,b,c,d, $(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad-bc)^2$, to rewrite $\prod[(x-b_j)^2 + c_j^2]^{l_j}$ as a sum of squares of polynomials.

The remaining $(c) \Rightarrow (a)$ is obvious.