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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 3

These exercises will be collected Tuesday 10 November in the mailbox number 15 of the Mathematics department.

Denote the ring of **real formal power series in one variable**, respectively **in several variables**, by:

$$\mathbb{R}[[X]] := \left\{ \sum_{i=0}^{\infty} a_i X^i \mid a_i \in \mathbb{R} \right\},$$
$$\mathbb{R}[[X_1,\dots,X_n]] := \left\{ \sum_{i=(i_1,\dots,i_n)\in\mathbb{N}^n} a_i X_1^{i_1}\cdots X_n^{i_n} \mid a_i \in \mathbb{R} \right\},$$

and the set of **real Laurent series** by:

$$\mathcal{K} := \left\{ \sum_{i=m}^{\infty} a_i X^i \mid m \in \mathbb{Z},\, a_i \in \mathbb{R} \right\}.$$

Definition 0.1 *The set of* ***real Puiseux series*** *is:*

$$\mathcal{P} := \bigcup_{N\in\mathbb{N}} \left\{ \sum_{i=m}^{\infty} a_i X^{i/N} \mid m \in \mathbb{Z},\, a_i \in \mathbb{R} \right\}.$$

The purpose of these exercises is to prove the following classical **Puiseux theorem**:
Theorem 0.2 (Puiseux theorem) *The set $\mathcal{P}$ is a real closed field.*

1. **Definition 0.3** *Define on $\mathcal{K}$:*
   - the ***termwise addition****:*
   $$\sum_{i=m}^{\infty} a_i X^i + \sum_{i=n}^{\infty} b_i X^i := \sum_{i=m}^{n-1} a_i X^i + \sum_{i=n}^{\infty} (a_i + b_i) X^i$$ *(where $m \leq n$ here as an example),*
   - the ***convolution product****:*
   $$\left(\sum_{i=m}^{\infty} a_i X^i\right)\left(\sum_{i=n}^{\infty} b_i X^i\right) := \sum_{i=m+n}^{\infty} \sum_{j+k=i} a_j b_k X^i.$$
   - the ***order relation****: for any $A(X) = \sum_{i=m}^{\infty} a_i X^i \in \mathcal{K}$,*
   $$A(X) \geq 0 \textit{ if and only if } A(X) = 0 \textit{ or the first coefficient } a_m \textit{ is positive.}$$

Show that $\mathcal{K}$ endowed with these relations is an ordered field.
(Hint: for the multiplicative inverse, recall Euler's formula

$$\frac{1}{1+U} = \sum_{i=0}^{\infty}(-1)^i U^i$$

for formal power series.)

2. Deduce that $\mathcal{P}$ is an ordered field.
(*Hint: extend the preceding relations, noting that $\mathcal{P}$ is a countable union of Laurent series fields*).

3. Consider a polynomial equation

$$P(X,Y) = A_0(X)Y^n + A_1(X)Y^{n-1} + \cdots + A_{n-1}(X)Y + A_n(X) = 0$$

with coefficients in $\mathcal{P}$. Show that, up to some changes of the variable X and of the unknown Y, one can reduce without loss of generality to an equation where $A_0(X) = 1$ (**unitary** polynomial) with coefficients in $\mathbb{R}[[X]]$.

We consider known the following result:

Lemma 0.4 *Given a field K, the polynomial ring $K[Y]$ is a principal ideal domain and any two polynomials $P(Y)$ and $Q(Y)$ have a greatest common divisor $D(Y)$ expressible in the form*

$$D(Y) = A(Y)P(Y) + B(Y)Q(Y).$$

In particular, if $P_0(Y)$ and $Q_0(Y)$ are relatively prime polynomials, then there are polynomials $A_0(Y)$ and $B_0(Y)$ such that

$$1 = A_0(Y)P_0(Y) + B_0(Y)Q_0(Y).$$

4. Given two relatively prime polynomials $P(Y)$ and $Q(Y)$ of degree p and q, show that for any polynomial $F(Y)$ of degree strictly less than $p+q$, there exist polynomials $A(Y)$ and $B(Y)$ of degree less than q and p respectively, such that

$$F(Y) = A(Y)P(Y) + B(Y)Q(Y).$$

(*Hint: use the cited lemma and the euclidean division on $F(Y)A_0(Y)$ and $F(Y)B_0(Y)$*).

The following technical result relies essentially on the preceding result and the Inverse Function theorem. We suppose it known for the purpose of this exercise.

Lemma 0.5 *Let $a := (a_1, \ldots, a_n) \in \mathbb{R}^n$ and*

$$F(X_1, \ldots, X_n, Y) = Y^n + X_1 Y^{n-1} + \cdots + X_{n-1}Y + X_n.$$

Assume that $F_0(Y) := F(a_1, \ldots, a_n, Y)$ can be written as the product of two relatively prime factors $P_0(Y)$ and $Q_0(Y)$ of degrees $p \geq 1$ and $q \geq 1$, respectively. Then there are series $C_1(X_1, \ldots, X_n), \ldots, C_p(X_1, \ldots, X_n)$ and $D_1(X_1, \ldots, X_n), \ldots, D_q(X_1, \ldots, X_n)$ in $\mathbb{R}[[X_1, \ldots, X_n]]$ such that

$$\begin{aligned} P(X_1, \ldots, X_n, Y) &:= Y^p + C_1(X_1, \ldots, X_n)Y^{p-1} + \cdots + C_p(X_1, \ldots, X_n) \\ Q(X_1, \ldots, X_n, Y) &:= Y^p + D_1(X_1, \ldots, X_n)Y^{p-1} + \cdots + D_q(X_1, \ldots, X_n) \end{aligned}$$

satisfy

$$F(X_1,\ldots,X_n,Y) = P(X_1,\ldots,X_n,Y)Q(X_1,\ldots,X_n,Y),$$

and

$$P(a_1,\ldots,a_n,Y) = P_0(Y),\ \ Q(a_1,\ldots,a_n,Y) = Q_0(Y).$$

5. **Hensel's lemma**. Let $F(X,Y)$ be a polynomial in Y of the form

$$F(X,Y) = Y^n + A_1(X)Y^{n-1} + \cdots + A_n(X),$$

where each $A_i \in \mathbb{R}[[X]]$. Suppose that $F(0,Y)$ factors into the product of relatively prime real factors $P_0(Y)$ and $Q_0(Y)$ of degrees p and q, respectively. Show that $F(X,Y)$ factors into the product of $P(X,Y)$ and $Q(X,Y)$ of same degrees p and q respectively, with coefficients in $\mathbb{R}[[X]]$ and for which

$$P(0,Y) = P_0(Y),\ Q(0,Y) = Q_0(Y).$$

6. (a) Deduce that any series $A(X) \in \mathbb{R}[[X]]$ with $A(X) > 0$ has a square root in $\mathcal{P}$. (*Hint: consider the equation* $Y^2 - A(X) = 0$).

(b) Deduce also that any polynomial

$$F(X,Y) = Y^n + A_1(X)Y^{n-1} + \cdots + A_n(X)$$

with degree n that is odd has a root in $\mathcal{P}$.
(*Hint: proceed by induction on* p *where* $n = 2p + 1$ *and use the fact that* $\mathbb{R}$ *is real closed*).

7. Conclude that $\mathcal{P}$ is real closed.
(*Hint: use criterion* (*iii*) *of Artin-Schreier's theorem and question 3*).