Universität Konstanz
Fachbereich Mathematik und Statistik
Prof. Dr. Salma Kuhlmann
Mitarbeiter: Dr. Mickaël Matusinski
Büroraum F 409
mickael.matusinski@uni-konstanz.de

Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 15

1. **Definition 0.1** *Let* $(G, +, \leq)$ *be an ordered abelian group. For any* $x \in G$, $x \neq 0$, *we define*
$$C_x := \bigcap\{C \text{ convex subgroup of } G,\ x \in C\}.$$
This is the smallest convex subgroup of G which contains x.
We also denote
$$D_x := \bigcup\{C \text{ convex subgroup of } G,\ x \notin C\}.$$
Prove the following proposition:
Proposition 0.2 *(a)* D_x *is the biggest convex subgroup of G which does not contain x.*
(b) The extension from D_x *to* C_x *is a* ***jump*** *(=* ***Sprung****), i.e. for any* $D_x \subseteq C \subseteq C_x$ *with C convex, then* $C = D_x$ *or* $C = C_x$*. We write* $D_x \prec C_x$.
(c) Consequently, the ordered abelian group $B_x := C_x/D_x$ *has no proper non trivial convex subgroup.*

2. **Definition 0.3** *An ordered abelian group* $(A, +, \leq)$ *is said to be* ***archimedian*** *if for any* $a_1,a_2 \in A$ *with* $a_1 \neq 0$ *and* $a_2 \neq 0$*, there exists* $n \in \mathbb{N}$ *such that* $n|a_1| \geq |a_2|$ *and* $n|a_2| \geq |a_1|$ *(where* $|a| := \max\{a; -a\}$*).*

   Prove the following proposition:
   Proposition 0.4 *An ordered abelian group* $(A, +, \leq)$ *is archimedian if and only if it has no non trivial proper convex subgroup.*

3. **Definition 0.5** • *Given an ordered abelian group* $(G, +, \leq)$*, two nonzero elements* $x,y \in G$ *are said to be* ***archimedian equivalent****, denoted by* $x \sim^+ y$*, if there exists* $n \in \mathbb{N}$ *such that* $n|x| \geq |y|$ *and* $n|y| \geq |x|$.
• *Otherwise, given two nonzero elements* $x,y \in G$*, if we have* $n|x| < |y|$ *for any* $n \in \mathbb{N}$*, then we denote* $x <<^+ y$.

Prove the following proposition:

Proposition 0.6 *The relation $\sim^+$ is compatible with the relation $<<^+$ in the following sense: for any nonzero $x,y,z \in G$,*

$$\text{if } x <<^+ y \text{ and } z \sim^+ x, \text{ then } z <<^+ y;$$
$$\text{if } x <<^+ y \text{ and } z \sim^+ y, \text{ then } x <<^+ z.$$

4. Given an ordered abelian group $(G, +, \leq)$, we consider the set $\Gamma := G \setminus \{0\}/\sim^+$ of its archimedean equivalence classes. We define a relation on Γ by, for any nonzero $x,y \in G$,

$$[y] <_\Gamma [x] \Leftrightarrow x <<^+ y.$$

Prove the following proposition:

Proposition 0.7 *(a) The relation $\leq_\Gamma$ is a total ordering on Γ.*
*The ordered set $(\Gamma := G \setminus \{0\}/\sim^+, \leq_\Gamma)$ is called the **rank** of G, denoted by $Rank(G)$.*

(b) For any nonzero $x \in G$, denote its archimedean equivalence class $[x] := v(x)$, and denote $[0] := \infty$. The map

$$\begin{array}{cccc} v: & G & \to & \Gamma \cup \{\infty\} \\ & x & \mapsto & v(x) \end{array}$$

*is a valuation, which is called the **natural valuation** of G.*

Definition 0.8 *Let $(\Gamma, \leq)$ be an ordered set and $\{B_\gamma, \gamma \in \Gamma\}$ be a family of archimedean abelian groups (consequently $B_\gamma \hookrightarrow (\mathbb{R}, +, \leq)$ by Hölder's theorem). The **ordered Hahn sum** is defined to be the Hahn sum $G = \overrightarrow{\coprod}_{\gamma\in\Gamma} B_\gamma$ (i.e. the direct sum from the B_γ's) endowed with the lexicographic ordering. Similarly, we define the **ordered Hahn product** $\overrightarrow{\mathbf{H}}_{\gamma\in\Gamma} B_\gamma$.*

(c) Given $x \in G$, $x \neq 0$, we put $v(x) := \gamma \in \Gamma$. Then we have

$$\begin{array}{rcl} G^\gamma := \{a \in G \mid v(a) \geq \gamma\} & = & C_x; \\ G_\gamma := \{a \in G \mid v(a) > \gamma\} & = & D_x. \end{array}$$

and consequently

$$G^\gamma/G_\gamma =: B(\gamma) = B_x := C_x/D_x$$

is an archimedean group.

(Hint: prove that for any nonzero $x,y \in G$, we have $x \sim^+ y \Leftrightarrow C_x = C_y$ and $D_x = D_y$.)