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**Übungen zur Vorlesung Reelle algebraische Geometrie**

**Blatt 13**

*These exercises will be collected Tuesday 2 February in the mailbox number 15 of the Mathematics department.*

*All the exercises are devoted to the new chapter on valuation theory. We will resume the chapter on dimension of semi algebraic sets in the next semester with more detailed lectures and exercises.*

1. **Definition 0.1** *Let $(M,v)$ be a **valued (=bewerte) module** and $\Gamma = v(M \setminus \{0\})$ its value set. Für any $\gamma \in \Gamma$, we define $M^\gamma := \{x \in M \mid v(x) \geq \gamma\}$, $M_\gamma := \{x \in M \mid v(x) > \gamma\}$ which are submodules of $M$, and $B(M,\gamma) := M^\gamma / M_\gamma$ which is also a module.*
   *The system of modules $S(M) := [\Gamma, \{B(M,\gamma),\ \gamma \in \Gamma\}]$ is called the **skeleton** of $(M,v)$.*
   The aim of this exercise is to prove the following lemma:
   **Lemma 0.2** *The skeleton is an isomorphism invariant, i.e. if two valued modules $(M_1,v_1)$, $(M_2,v_2)$ are isomorphic, then so are the corresponding two skeletons $S(M_1)$, $S(M_2)$.*
   (a) Let $h : (M_1,v_1) \to (M_2,v_2)$ be an isomorphism of valued modules, and $\Gamma_1 := v_1(M_1 \setminus \{0\})$ and $\Gamma_2 := v_2(M_2 \setminus \{0\})$ the corresponding value sets. Show that the map
   $$\tilde{h} : \Gamma_1 \to \Gamma_2,\ \tilde{h}(v_1(x)) := v_2(h(x))$$
   is well-defined and is an isomorphism of ordered sets.

   (b) For any $\gamma \in \Gamma_1$, we define
   $$\begin{array}{rccc} h_\gamma : & B(M_1,\gamma) & \to & B(M_2,\tilde{h}(\gamma)) \\ & \Pi^{M_1}(\gamma,x) & \mapsto & \Pi^{M_2}(\tilde{h}(\gamma),h(x)). \end{array}$$

Show that this map is well-defined for any $\gamma \in \Gamma_1$, and is an isomorphism of modules. Then deduce the desired Lemma 0.2.

2. **Definition 0.3** *Consider a* ***system of torsion free modules*** $S = [\Gamma, \{B(\gamma); \gamma \in \Gamma\}]$, *and denote by* $\prod_{\gamma\in\Gamma} B(\gamma)$ *the corresponding product module. Denote by* $\bigoplus_{\gamma\in\Gamma} B(\gamma)$ *the submodule of maps* $s \in \prod_{\gamma\in\Gamma} B(\gamma)$ *with finite support, and* $\coprod_{\gamma\in\Gamma} B(\gamma)$ *the* ***Hahn sum*** *of* $S$*, i.e. the valued module* $(\bigoplus_{\gamma\in\Gamma} B(\gamma), v_{\min})$ *where* $v_{\min}(s) := \min(\text{support } s)$ *for all* $s \in \bigoplus_{\gamma\in\Gamma} B(\gamma) \setminus \{0\}$.

   *Denote by* $\boldsymbol{H}_{\gamma\in\Gamma} B(\gamma)$ *the* ***Hahn product*** *of* $S$*, i.e. the submodule of maps* $s \in \prod_{\gamma\in\Gamma} B(\gamma)$ *with well-ordered support, also equipped with the valuation* $v_{\min}$.

   (a) Show that $\coprod_{\gamma\in\Gamma} B(\gamma)$ is a valued submodule of $\mathbf{H}_{\gamma\in\Gamma} B(\gamma)$.

   (b) Show that $S\left(\coprod_{\gamma\in\Gamma} B(\gamma)\right) = S = S\left(\mathbf{H}_{\gamma\in\Gamma} B(\gamma)\right)$.

3. **Definition 0.4** *Let* $(\Gamma, \leq)$ *be a totally ordered set. We say that* $\Gamma$ *is* ***well-ordered*** *if any non empty subset* $A \subseteq \Gamma$ *has a smallest element.*
   *Given a well-ordered set* $(\Gamma, \leq)$*, its* ***order type*** $ot(\Gamma)$ *is defined to be a fixed representative of its equivalence class by ordered set isomorphism, and is called an* ***ordinal number****. In particular, the order type of the set of natural numbers is denoted by* $ot(\mathbb{N}) := \omega$*. It is the smallest infinite ordinal number.*

   (a) Given 2 ordered sets $(A, \leq)$ and $(B, \leq)$, one defines the **sum of ordered sets**: $(A, \leq_A) + (B, \leq_B) = A + B := (A \sqcup B, \leq_+)$ ($\sqcup$ = disjoint union) such that for any

   $$c_1, c_2 \in A \sqcup B \ , \ c_1 \leq_+ c_2 \Leftrightarrow \begin{cases} \text{either } c_1, c_2 \in A \text{ and } c_1 \leq_A c_2 \\ \text{or } (c_1 \in A,\ c_2 \in B) \\ \text{or } (c_1, c_2 \in B \text{ and } c_1 \leq_B c_2). \end{cases}.$$

   Show that, if $A$ and $B$ are well-ordered, then so is $A + B$.

   (b) Suppose that $A$ and $B$ are well-ordered sets. Denote $\alpha := ot(A)$ and $\beta := ot(B)$. One defines the **sum of ordinals** as

   $$\alpha + \beta := ot(A + B).$$

   Show that the sum of ordinals is well-defined and not commutative.
   (Hint: compute $1 + \omega$ and $\omega + 1$.)

   (c) For any $n \in \mathbb{N}$, we put $\omega.n := \omega + \cdots + \omega$ ($n$ times). For any $n \in \mathbb{N}$, give an example of a subset $Q_n$ of $\mathbb{Q}$ such that $ot(Q_n) = \omega.n$.

(d) Given 2 ordered sets $(A, \leq_A)$ and $(B, \leq_B)$, one defines the **product of ordered sets**:

$$(B, \leq_B).(A, \leq_A) = B.A := (A \times B, \leq_{\text{lex}}) \text{ such that } \leq_{\text{lex}} \text{ is the lexicographic ordering, i.e. for any } (a_1,b_1),(a_2,b_2) \in A \times B,$$

$$(a_1,b_1) \leq_{\text{lex}} (a_2,b_2) \Leftrightarrow \begin{cases} a_1 <_A a_2 \\ \text{or } a_1 = a_2 \text{ and } b_1 \leq_B b_2 \end{cases}.$$

Show that, if $A$ and $B$ are well-ordered, then so is $A.B$.

(e) Suppose that $A$ and $B$ are well-ordered sets. Denote $\alpha := ot(A)$ and $\beta := ot(B)$, one defines the **product of ordinals**:

$$\alpha.\beta := ot(A.B).$$

Show that the product of ordinals is well-defined and not commutative.
(Hint: compute $2.\omega$ and $\omega.2$.)

(f) For any $n \in \mathbb{N}$, we put $\omega^n := \omega. \cdots .\omega$ ($n$ times). For any $n \in \mathbb{N}$, give an example of a subset $Q_n$ of $\mathbb{Q}$ such that $ot(Q_n) = \omega^n$.