Universität Konstanz
Fachbereich Mathematik und Statistik
Prof. Dr. Salma Kuhlmann
Mitarbeiter: Dr. Mickaël Matusinski
Büroraum F 409
mickael.matusinski@uni-konstanz.de

Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 11

These exercises will be collected Tuesday 19 January in the mailbox number 15 of the Mathematics department.

Theorem 0.1 (Curve Selection Lemma) *Let R be a real closed field. Let A be a semi-algebraic subset of R^n and $\underline{x} \in R^n$ a point belonging to $\overline{A}$, the closure of A. Then there exists a continuous semi-algebraic map*

$$f : [0,1] \to R^n$$

such that $f(0) = \underline{x}$ and $f(]0,1]) \subset A$.

Definition 0.2 *A polynomial $f(\underline{X},Y) \in R[\underline{X},Y]$ is said to be **quasi-monic** with respect to Y if*

$$f(\underline{X},Y) = a_d Y^d + g_{d-1}(\underline{X})Y^{d-1} + \cdots + g_0(\underline{X}),$$

where a_d is a nonzero element of R.

The Curve Selection Lemma is proved together with the following lemma:

Lemma 0.3 *Denote $\underline{X} = (X_1,\ldots,X_n)$. Let $f_1,\ldots,f_s$ be a family of polynomials in $R[\underline{X},Y]$. Suppose that the family is **stable under derivation** with respect to Y and that all f_k are **quasi-monic** with respect to Y. Let $(A_i,(\xi_{i,j})_{j=1,\ldots,l_i})_{i=1,\ldots,m}$ be a slicing of $f_1,\ldots,f_s$. Then every function $\xi_{i,j}$ can be continuously extended to $\overline{A_i}$.*

1. This exercise deals with two claims used in the case (*iii*) of the proof of the Curve Selection Lemma and the Lemma 0.3. Namely, the case (*iii*) is the one where we suppose that the Curve Selection Lemma and the Lemma 0.3 hold for some $n \in \mathbb{N}$, and where we prove that the Curve Selection Lemma holds for $n+1$. Therefore, we consider a point $(\underline{x},y) \in \overline{A}$ in the closure of some semi-algebraic subset $A \subset R^{n+1}$. Let $f_1,\ldots,f_s \in R[\underline{X},Y]$ be a family of non trivial polynomials defining A (as a boolean combination of equations and inequalities).

   (a) (i) Show that any non trivial polynomial in $R[\underline{X},Y]$ can be reduced to a quasi-monic one, using a linear change of variable of type

   $$X_i = X_i' + a_i Y, \quad a_i \in R, \quad i = 1,\ldots,n.$$

(ii) Deduce that, without loss of generality, we can suppose that the family $f_1, \dots, f_s$ defining A is quasi-monic and closed under derivation with respect to Y.

(b) Now we are concerned with the very last part of the proof of *(iii)*. We consider a slicing $(A_i, (\xi_{i,j})_{j=1,\dots,l_i})_{i=1,\dots,m}$ of $f_1, \dots, f_s$. During the lecture, we delt with the case where the point $(\underline{x}, y)$ is in the closure of a slice $]\xi_{i,j}, \xi_{i,j+1}[\subset A$, with $j = 1, \dots, l_i - 1$. By Lemma 0.3 for n, we noted that $\xi_{i,j}$ and $\xi_{i,j+1}$ can be extended continuously to $\underline{x}$. Now applying the CSL for n, there exists a curve $\phi : [0,1] \to R^n$ such that $\phi(0) = \underline{x}$ and $\phi(]0,1]) \subset A_i$. There was a subcase that we did not prove during the lecture: the one where $(\underline{x}, y) \in Cl(\Gamma(\xi_{i,j}))$ or $(\underline{x}, y) \in Cl(\Gamma(\xi_{i,j+1}))$ (these graphs are included in the closure of A_i, but may not be included in A).

(i) Check that the map

$$f = (\phi, \psi) : [0,1] \to R^{n+1} = R^n \times R$$

where

$$\forall t \in [0,1],\ \psi(t) := c[\frac{t}{2}(\xi_{i,j} \circ \phi)(t) + (1 - \frac{t}{2})(\xi_{i,j+1} \circ \phi)(t)]$$
$$+(1-c)[(1 - \frac{t}{2})(\xi_{i,j} \circ \phi)(t) + \frac{t}{2}(\xi_{i,j+1} \circ \phi)(t)]$$

$$\text{and} \quad c = \left| \begin{array}{ll} \dfrac{1}{2} & \text{if } \xi_{i,j}(\underline{x}) = \xi_{i,j+1}(\underline{x}) = y \\ \dfrac{y - \xi_{i,j}(\underline{x})}{\xi_{i,j+1}(\underline{x}) - \xi_{i,j}(\underline{x})} & \text{if } \xi_{i,j}(\underline{x}) < \xi_{i,j+1}(\underline{x}). \end{array} \right.$$

verifies the Curve Selection Lemma for $n+1$ for all $(\underline{x}, y) \in Cl(]\xi_{i,j}, \xi_{i,j+1}[)$.

(ii) Suppose now that the point $(\underline{x}, y)$ is in the closure of a slice $]\xi_{i,j}, \xi_{i,j+1}[\subset A$, where either $j = 0$ which means that the slice is $]-\infty, \xi_{i,1}[$, or $j = l_i$ which means that the slice is $]\xi_{i,l_i}, +\infty[$. Deduce that the Curve Selection Lemma holds for $n+1$ in this case and give an explicit formula for f as in the preceding question.

2. (a) Let $A \subset R^n$ be a semi-algebraic set and $f : A \to R^m$ be a semi-algebraic map. Show that there exists a semi-algebraic decomposition $A_1 \bigcup \cdots \bigcup A_k = A$ such that $f_{|A_i}$ is continuous for $i = 1, \dots, k$.

   (b) Let $I \subset R$ be an interval and $f : I \to R^m$ be a semi-algebraic map. Deduce that f is continuous at all but finitely many $x \in I$.

3. Show that any semi-algebraic subset $A \subset R^n$ is **semi-algebraically locally connected**, i.e. any $x \in A$ has an arbitrarily small semi-algebraically connected neighbourhood.

4. Consider a semi-algebraic subset $A \subset R^{n+1} = R^n \times R$, and the projection
$$\pi : R^{n+1} \to R^n,\ \pi(\underline{x}, t) = \underline{x}\ (\underline{x} \in R^n,\ t \in R).$$
Show that there exists a semi-algebraic map $f : \pi(A) \to R$ such that $(\underline{x}, f(\underline{x})) \in A$ when $\underline{x} \in \pi(A)$.