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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 11 - Solution

Theorem 0.1 (Curve Selection Lemma) *Let R be a real closed field. Let A be a semi-algebraic subset of R^n and $\underline{x} \in R^n$ a point belonging to $\overline{A}$, the closure of A. Then there exists a continuous semi-algebraic map*

$$f : [0,1] \to R^n$$

such that $f(0) = \underline{x}$ and $f(]0,1]) \subset A$.

Definition 0.2 *A polynomial $f(\underline{X},Y) \in R[\underline{X},Y]$ is said to be* ***quasi-monic*** *with respect to Y if*

$$f(\underline{X},Y) = a_d Y^d + g_{d-1}(\underline{X})Y^{d-1} + \cdots + g_0(\underline{X}),$$

where a_d is a nonzero element of R.

The Curve Selection Lemma is proved together with the following lemma:

Lemma 0.3 *Denote $\underline{X} = (X_1,\ldots,X_n)$. Let $f_1,\ldots,f_s$ be a family of polynomials in $R[\underline{X},Y]$. Suppose that the family is* ***stable under derivation*** *with respect to Y and that all f_k are* ***quasi-monic*** *with respect to Y. Let $(A_i,(\xi_{i,j})_{j=1,\ldots,l_i})_{i=1,\ldots,m}$ be a slicing of $f_1,\ldots,f_s$. Then every function $\xi_{i,j}$ can be continuously extended to $\overline{A_i}$.*

1. This exercise deals with two claims used in the case (iii) of the proof of the Curve Selection Lemma and the Lemma 0.3. Namely, the case (iii) is the one where we suppose that the Curve Selection Lemma and the Lemma 0.3 hold for some $n \in \mathbb{N}$, and where we prove that the Curve Selection Lemma holds for $n+1$. Therefore, we consider a point $(\underline{x},y) \in \overline{A}$ in the closure of some semi-algebraic subset $A \subset R^{n+1}$. Let $f_1,\ldots,f_s \in R[\underline{X},Y]$ be a family of non trivial polynomials defining A (as a boolean combination of equations and inequalities).

   (a) (i) Consider

   $$f(\underline{X},Y) = g_m(\underline{X})Y^m + \cdots + g_0(\underline{X})$$

   and $\underline{a} = (a_1,\ldots,a_n) \in R^n$. Denote for any $k = 1,\ldots,m$,

   $$g_k(\underline{X}) = \sum_{0 \le |I| \le d_k} c_{k,I}\underline{X}^I$$

with $I = (i_1, \dots, i_n) \in \mathbb{N}^n$, $|I| = i_1 + \dots + i_n$, $\underline{X}^I = X_1^{i_1}, \dots, X_n^{i_n}$, and d_k is the total degree of g_k.
Performing the following linear change of coordinates

$$\underline{X} = \underline{\tilde{X}} + \underline{a}Y$$

we obtain for any $k = 1, \dots, m$

$$\begin{aligned} g_k(\underline{X}) &= \tilde{g}(\underline{\tilde{X}}, Y) \\ &= \sum_{0 \le |I| \le d_k} c_{k,I}(\underline{\tilde{X}} + \underline{a}Y)^I \\ &= (\sum_{|I|=d_k} c_{k,I}\underline{a}^I)Y^{d_k} + \dots + g_k(\underline{\tilde{X}}) \end{aligned}$$

The leading coefficient $\sum_{|I|=d_k} c_{k,I}\underline{a}^I$ is a non trivial homogenous polynomial of degree d_k in the a_i's: : we can choose them so that it is non zero (take $\underline{a}$ in the complement in R^n of the algebraic set defined by this polynomial).

Now, denote $d := \max\{k + d_k; k = 1, \dots, m\}$. We obtain:

$$\begin{aligned} f(\underline{X}, Y) &= \tilde{f}(\underline{\tilde{X}}, Y) \\ &= \sum_{k=0}^{m} \tilde{g}_k(\underline{X}, Y)Y^k \\ &= \sum_{k=0}^{m} [(\sum_{|I|=d_k} c_{k,I}\underline{a}^I)Y^{k+d_k} + \dots + g_k(\underline{\tilde{X}})Y^k] \\ &= [\sum_{k+d_k=d}(\sum_{|I|=d_k} c_{k,I}\underline{a}^I)]Y^d + \dots \text{(terms with degree less than } d \text{ in } Y) \end{aligned}$$

The leading coefficient is a sum over k such that $k + d_k = d$, of non trivial homogenous polynomials in the a_i's of degree d_k. So it is itself a non trivial polynomial. Thus we can choose the a_i's so that it is non zero (as before, consider $\underline{a}$ in the complement in R^n of the algebraic set defined by this polynomial).

(ii) It suffices to note that the derivative with respect to Y of a quasi-monic polynomial is itself quasi-monic.

(b) Now we are concerned with the very last part of the proof of (*iii*). We consider a slicing $(A_i, (\xi_{i,j})_{j=1,\dots,l_i})_{i=1,\dots,m}$ of $f_1, \dots, f_s$. During the lecture, we delt with the case where the point $(\underline{x}, y)$ is in the closure of a slice $]\xi_{i,j}, \xi_{i,j+1}[\subset A$, with $j = 1, \dots, l_i - 1$. By Lemma 0.3 for n, we noted that $\xi_{i,j}$ and $\xi_{i,j+1}$ can be extended continuously to $\underline{x}$. Now applying the CSL for n, there exists a curve $\phi : [0,1] \to R^n$ such that $phi(0) = \underline{x}$ and $\phi(]0,1]) \subset A_i$. There was a subcase that we did not prove during the lecture: the one where $(\underline{x}, y) \in Cl(\Gamma(\xi_{i,j}))$ or $(\underline{x}, y) \in Cl(\Gamma(\xi_{i,j+1}))$ (these graphs are included in the closure of A_i, but may not be included in A).

(i) Consider the map

$$f = (\phi, \psi) : [0,1] \to R^{n+1} = R^n \times R$$

where

$$\forall t \in [0,1],\ \psi(t) := c[\frac{t}{2}(\xi_{i,j} \circ \phi)(t) + (1 - \frac{t}{2})(\xi_{i,j+1} \circ \phi)(t)]$$
$$+(1-c)[(1 - \frac{t}{2})(\xi_{i,j} \circ \phi)(t) + \frac{t}{2}(\xi_{i,j+1} \circ \phi)(t)]$$

$$\text{and} \quad k = \left| \begin{array}{ll} \frac{1}{2} & \text{if } \xi_{i,j}(\underline{x}) = \xi_{i,j+1}(\underline{x}) = y \\ \frac{y - \xi_{i,j}(\underline{x})}{\xi_{i,j+1}(\underline{x}) - \xi_{i,j}(\underline{x})} & \text{if } \xi_{i,j}(\underline{x}) < \xi_{i,j+1}(\underline{x}). \end{array} \right.$$

It suffices to check that $f(0) := (\phi(0),\psi(0)) = (\underline{x},y)$ and that $f(t) := (\phi(t),\psi(t)) \in]\xi_{i,j},\xi_{i,j+1}[)$ for all $t \in]0,1]$.

(ii) Suppose now that the point $(\underline{x},y)$ is in the closure of a slice $]\xi_{i,j},\xi_{i,j+1}[\subset A$, where either $j = 0$ which means that the slice is $]-\infty,\xi_{i,1}[$, or $j = l_i$ which means that the slice is $]\xi_{i,l_i}, +\infty[$.

Consider for instance the case $j = 0$, i.e. $(\underline{x},y) \in Cl(]-\infty,\xi_{i,1}[)$ with $]-\infty,\xi_{i,1}[\subset A$. By Lemma 0.3 for n, note that $\xi_{i,1}$ can be extended continuously to $\underline{x}$. We put $\tilde{\xi}_{i,0} := \xi_{i,1} - d$ where for example $d := 1 + (\xi_{i,1}(\underline{x}) - y)$. Then $(\underline{x},y) \in Cl(]\tilde{\xi}_{i,0},\xi_{i,1}[)$ and $]\tilde{\xi}_{i,0},\xi_{i,1}[\subset A$. Now, we can use the preceding result. Namely, we define

$$f = (\phi,\psi) : [0,1] \to R^{n+1} = R^n \times R$$

where

$$\forall t \in [0,1],\ \psi(t) := c[\frac{t}{2}(\tilde{\xi}_{i,0} \circ \phi)(t) + (1 - \frac{t}{2})(\xi_{i,1} \circ \phi)(t)]$$
$$+(1-c)[(1 - \frac{t}{2})(\tilde{\xi}_{i,0} \circ \phi)(t) + \frac{t}{2}(\xi_{i,1} \circ \phi)(t)]$$

$$\text{and} \quad c = \left| \begin{array}{ll} \frac{1}{2} & \text{if } \tilde{\xi}_{i,0}(\underline{x}) = \xi_{i,1}(\underline{x}) = y \\ \frac{y - \tilde{\xi}_{i,0}(\underline{x})}{\xi_{i,1}(\underline{x}) - \tilde{\xi}_{i,0}(\underline{x})} & \text{if } \tilde{\xi}_{i,0}(\underline{x}) < \xi_{i,1}(\underline{x}). \end{array} \right.$$

We obtain:

$$\psi(t) = (\xi_{i,1} \circ \phi)(t) + (\frac{1}{2} - c)dt - d(1-c).$$

The case for which $j = l_i$ is similar.

2. (a) Let $A \subset R^n$ be a semi-algebraic set and $f : A \to R^m$ be a semi-algebraic map. For any $k = 1,\ldots,m$, denote by $\pi_k : R^m \to R$ the projection onto the k^{th} component of R^m, and f_k the semialgebraic map $f_k := \pi_k \circ f : A \to R$. The map f is continuous at some $\underline{x} \in A$ if and only if f_k is continuous at $\underline{x}$ for $k = 1,\ldots,m$. For any $k = 1,\ldots,m$, consider the graph $\Gamma(f_k) := \{(\underline{x},y) \in A \times R \mid y = f_k(\underline{x})\}$ which is semialgebraic, and a slicing of it $(A_i^{(k)},\{\xi_{i,j}^{(k)}, j = 1,\ldots,l_{k,i}\})_{i=1,\ldots,n_k}$. From the Theorem of Cellular Decomposition, $\Gamma(f_k)$ is the finite union of $\Gamma(\xi_{i,j}^{(k)})$ and slices $]\xi_{i,j}^{(k)},\xi_{i,j+1}^{(k)}[$. But since $\Gamma(f_k)$ is a graph, it has empty interior. So it is only a finite union of graphs of $\xi_{i,j}^{(k)}$'s. More precisely, for each $i = 1,\ldots,n_k$, we have $f_k(\underline{x}) = \xi_{i,j}^{(k)}(\underline{x})$ for all $\underline{x} \in A_i^{(k)}$, for some fixed $j = 1,\ldots,l_{k,i}$. Now, we notice that f_k is continuous on each $A_i^{(k)}$ since the corresponding $\xi_{i,j}^{(k)}$ is so.

To conclude it suffices to consider the decomposition $(A_i)_i$ of A which is the intersection of all the decompositions $(A_i^{(k)})_{i=1,\dots,n_k}$: f is then continuous on each of the A_i's.

(b) From the preceding result, there exists a semi-algebraic decomposition $I = I_1 \bigcup \cdots \bigcup I_m$ such that $f_{|I_k}$ is continuous. Then, notice that semi-algebraic subsets of R are finite unions of intervals and points.

3. Consider a semi-algebraic subset $A \subset R^n$, and an element $x \in A$. Consider a semi-algebraic neighbourhood U of x in A. Let U_0 be the semi-algebraic connected component of U which contains x. Then U_0 is open in U and is a semi-algebraically connected neighbourhood of x in U.

4. For $\underline{x} \in R^n$, let $A_{\underline{x}} := \{t \in R \mid (\underline{x},y) \in A\}$. Since $A_{\underline{x}}$ is semi-algebraic in R, it is a finite union of points and intervals. For any $\underline{x} \in \pi(A)$, $A_{\underline{x}}$ is nonempty. We define $f(\underline{x})$ by:

   (a) if $A_{\underline{x}} = R$, let $f(\underline{x}) := 0$;

   (b) if $A_{\underline{x}}$ has a least element t_0, let $f(\underline{x}) := t_0$;

   (c) if the leftmost interval of $A_{\underline{x}}$ is $]t_0,t_1[$, let $f(\underline{x}) := \dfrac{t_0 + t_1}{2}$;

   (d) if the leftmost interval of $A_{\underline{x}}$ is $]-\infty,t_0[$, let $f(\underline{x}) := t_0 - 1$;

   (e) if the leftmost interval of $A_{\underline{x}}$ is $]t_0, +\infty[$, let $f(\underline{x}) = t_0 + 1$.
   This exhausts all possibilities. Clearly, f is semi-algebraic and $(\underline{x}, f(\underline{x})) \in A$ when $\underline{x} \in \pi(A)$.