LINEARE ALGEBRA II: SOLUTIONS TO SHEET 12

Question 1:

(a) Let V be a complex inner product space (hermitescher Raum).
From UB11 question 11.4(a)(iv), we know that if $\mathcal{B}$ is an orthonormal basis for V then

$$[T^*]_{\mathcal{B}} := \overline{([T]_{\mathcal{B}})^t}.$$

(i) $\Rightarrow$ (iii): Suppose that T is unitary (unitär). Since $TT^* = I$,

$$\begin{aligned} [T]_{\mathcal{B}}[T^*]_{\mathcal{B}} &= [TT^*]_{\mathcal{B}} \\ &= [I]_{\mathcal{B}} \\ &= I \end{aligned}$$

Thus $[T]_{\mathcal{B}}\overline{([T]_{\mathcal{B}})^t} = I$. So $[T]_{\mathcal{B}}$ is a unitary matrix.
(iii) $\Rightarrow$ (ii) is clear.
(ii) $\Rightarrow$ (i) Suppose that $\mathcal{B}$ is an orthonormal basis such that $[T]_{\mathcal{B}}\overline{[T]^t_{\mathcal{B}}} = I$. Thus $[TT^*]_{\mathcal{B}} = [T]_{\mathcal{B}}[T^*]_{\mathcal{B}} = [T]_{\mathcal{B}}\overline{[T]^t_{\mathcal{B}}} = I$. Thus $[TT^*]_{\mathcal{B}} = I$. So $TT^* = I$.

(b) The proof is identical to the one above except that if $\mathcal{B}$ is an orthonormal basis for V then $[T^*]_{\mathcal{B}} = [T]^t_{\mathcal{B}}$.

Question 2: Since T is normal, there exists an orthonormal basis for V consisting of eigenvectors of T. Let $\{b_1, ..., b_n\}$ be such a basis and let $Tb_i = \lambda_i b_i$ for $1 \leq i \leq n$.

(a) Suppose T is a normal operator on V, a complex inner product space, such that all eigenvalues of T are real and positive. By korollar 4 vorlesung 23, T is hermitian. So it remains to show that T is positive.

Take $x \in V$ and suppose that $r_1, ..., r_n \in \mathbb{C}$ are such that $x = \sum_{i=1}^n r_i b_i$. Thus

$$\begin{aligned}(Tx|x) &= (T(\sum_{i=1}^n r_i b_i)|\sum_{j=1}^n r_j b_j) \\ &= (\sum_{i=1}^n r_i T b_i|\sum_{j=1}^n r_j b_j) \\ &= (\sum_{i=1}^n r_i \lambda_i b_i|\sum_{j=1}^n r_j b_j) \\ &= \sum_{i=1}^n r_i \lambda_i (b_i|\sum_{j=1}^n r_j b_j) \\ &= \sum_{i=1}^n r_i \lambda_i (\sum_{j=1}^n \overline{r_j}(b_i|b_j)).\end{aligned}$$

Since $(b_i|b_j) = \delta_{ij}$,

$$\begin{aligned}\sum_{i=1}^n r_i \lambda_i (\sum_{j=1}^n \overline{r_j}(b_i|b_j)) &= \sum_{i=1}^n r_i \overline{r_i} \lambda_i (b_i|b_i) \\ &= \sum_{i=1}^n r_i \overline{r_i} \lambda_i\end{aligned}$$

which is greater than or equal to zero since $r_i\overline{r_i} \geq 0$ and $\lambda_i \geq 0$ for all $1 \leq i \leq n$. Thus T is positive.

For the other direction: Since T is hermitian, all eigenvalues are real. Suppose that $Tv = \lambda v$ and $v \neq 0$. Then $\lambda(v|v) = (Tv|v) \geq 0$. Since $v \neq 0$, $(v|v) \neq 0$. Thus $\lambda \geq 0$.

(b) Again by korollar 4 vorlesung 23, T is hermitian. Take non-zero $x \in V$ and suppose that $r_1, ..., r_n \in \mathbb{C}$ are such that $x = \sum_{i=1}^n r_i b_i$. Thus

$$(Tx, x) = \sum_{i=1}^n r_i \overline{r_i} \lambda_i.$$

Since x is non-zero and $b_1, ..., b_n$ is a basis for V, at $r_i \neq 0$ for some $1 \leq i \leq n$. Thus $r_i\overline{r_i} > 0$ for some $1 \leq i \leq n$. Thus $\lambda_i r_i \overline{r_i} > 0$. Thus $(Tx|x) > 0$.

For the other direction: Since T is hermitian, all eigenvalues are real. Suppose that $Tv = \lambda v$ and $v \neq 0$. Then $\lambda(v, v) = (Tv, v) > 0$. Since $v \neq 0$, $(v|v) \neq 0$. Thus $\lambda > 0$.

(c) This statement holds more generally. Let V be a finite dimensional vector space over a field K. Since V is finite dimensional, T is invertible if and only if $\ker T = \{0\}$ if and only if for all $v \in V$, $Tv = 0$ implies $v = 0$ if and only if zero is not an eigenvalue of T.

(d) Suppose that T is idempotent $v \in V$ is non-zero and that $Tv = \lambda v$. Then $T^2 v = \lambda^2 v$ and $\lambda v = Tv = T^2 v$. Thus $\lambda^2 v = \lambda v$. So $(\lambda^2 - \lambda)v = 0$. Thus $\lambda^2 - \lambda = 0$. So $\lambda(\lambda - 1) = 0$. Thus $\lambda = 0$ or $\lambda = 1$.

Suppose T is a normal operator and all eigenvalues of T are either zero or one. If $Tb_i = 0$ then $T^2 b_i = 0$ and if $Tb_i = b_i$ then $T^2 b_i = b_i$. Thus, for all $1 \leq i \leq n$, $Tb_i = T^2 b_i$. Thus, for all $1 \leq i \leq n$, $(T - T^2)b_i = 0$. So $T - T^2 = 0$, since $T - T^2$ is linear and a linear map which is zero on a basis is zero.

Question 3:

(a) Let V be a complex inner product space. Suppose T is normal. Let

$$T_1 := \frac{T + T^*}{2}$$

and

$$T_2 := \frac{iT^* - iT}{2}.$$

Then

$$T_1^* = \frac{(T + T^*)^*}{2} = \frac{T^* + T}{2} = T_1$$

and

$$T_2^* = \frac{(iT^* - iT)^*}{2} = \frac{-iT + iT^*}{2} = T_2.$$

Since $TT^* = T^*T$,

$$\begin{aligned} T_1 T_2 &= \left(\frac{T + T^*}{2}\right)\left(\frac{iT^* - iT}{2}\right) \\ &= \frac{iTT^* - iTT + iT^*T^* - iT^*T}{4} \\ &= \frac{iT^*T - iTT + iT^*T^* - iTT^*}{4} \\ &= \left(\frac{iT^* - iT}{2}\right)\left(\frac{T + T^*}{2}\right) \\ &= T_2 T_1. \end{aligned}$$

Suppose $T = T_1 + iT_2$, T_1, T_2 are hermitian and $T_1T_2 = T_2T_1$. Then $T^* = T_1^* - iT_2^*$ and since T_1, T_2 are hermitian $T_1^* - iT_2^* = T_1 - iT_2$. A simple computation now show that$TT^* = T^*T$.

(b) Example 1: Let T have matrix representation

$$\begin{pmatrix} i & 0 \\ 0 & 2 \end{pmatrix}$$

with respect to an orthonormal basis $\mathcal{B}$. Then

$$[T^*]_{\mathcal{B}} = \begin{pmatrix} -i & 0 \\ 0 & 2 \end{pmatrix}.$$

Since $[T]_{\mathcal{B}} \neq [T^*]_{\mathcal{B}}$, T is not hermitian. Since $[T]_{\mathcal{B}} \neq [-T^*]_{\mathcal{B}}$, T is not skew-hermitian. Since

$$[TT^*]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix},$$

T is not unitary.

Example 2: Let $T_1 = I$ and $T_2 = I$. Then $T = T_1 + iT_2$ is normal and $T^* = T_1 - iT_2$.So $T \neq T^*$, $T^* \neq -T$ and $TT^* = 2$.

Question 4: Since T is normal, there exists an orthonormal basis $\mathcal{B}$ of V such that

$$[T]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 & 0 & & & \\ 0 & \alpha_2 & & & \\ & & \ddots & & \\ & & & \alpha_{n-1} & 0 \\ & & & 0 & \alpha_n \end{pmatrix}$$

for some $\alpha_1, ..., \alpha_n \in \mathbb{C}$.
Thus

$$[T^*]_{\mathcal{B}} = \begin{pmatrix} \overline{\alpha_1} & 0 & & & \\ 0 & \overline{\alpha_2} & & & \\ & & \ddots & & \\ & & & \overline{\alpha_{n-1}} & 0 \\ & & & 0 & \overline{\alpha_n} \end{pmatrix}.$$

Note that $\alpha_i = \alpha_j$ implies $\overline{\alpha_i} = \overline{\alpha_j}$. Thus, using the Lagrange interpolation formula, there exists a polynomial $f \in \mathbb{C}[x]$ such that

$$f(\alpha_i) = \overline{\alpha_i}$$

for all $1 \leq i \leq n$.

Hence

$$f([T]_{\mathcal{B}}) = \begin{pmatrix} f(\alpha_1) & 0 & & & \\ 0 & f(\alpha_2) & & & \\ & & \ddots & & \\ & & & f(\alpha_{n-1}) & 0 \\ & & & 0 & f(\alpha_n) \end{pmatrix} = \begin{pmatrix} \overline{\alpha_1} & 0 & & & \\ 0 & \overline{\alpha_2} & & & \\ & & \ddots & & \\ & & & \overline{\alpha_{n-1}} & 0 \\ & & & 0 & \overline{\alpha_n} \end{pmatrix} = [T^*]_{\mathcal{B}}.$$

So $[f(T)]_{\mathcal{B}} = f([T]_{\mathcal{B}}) = [T^*]_{\mathcal{B}}$. So $f(T) = T^*$.