A dichotomy for T-convex fields with a monomial group

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The Dichotomy

For o-minimal fields $\mathcal{R} \models T$, expanded by a T-convex valuation ring and a monomial group:

- If T is power bounded, then this expansion of $\mathcal{R}$ is well-behaved.
- If $\mathcal{R}$ defines an exponential function, then $\mathbb{N}$ is externally definable.

o-minimal Preliminaries

Let T be a complete o-minimal theory extending the theory of $(\mathbb{R}; 0, 1, <, +, -, \cdot)$ in an appropriate language $\mathcal{L}$ and $\mathcal{R}$ be a model of T.

A **power function** is a function that looks like $f : x \mapsto x^\lambda$, where $\lambda := f'(1) \in \mathcal{R}$. These λ form the (sub)**field** Λ **of exponents of** $\mathcal{R}$. By Miller's dichotomy [Mil96], either $\mathcal{R}$ is

- **power bounded**, i.e. definable functions are eventually bounded by a power function, or
- $\mathcal{R}$ defines an **exponential function** i.e. $\exp\colon \mathcal{R} \xrightarrow{\sim} \mathcal{R}^{>}$ with $\exp = \exp'$.

T-convex Fields

A ***T*-convex subring** $\mathcal{O}$ of $\mathcal{R}$ is a convex subset of $\mathcal{R}$ which is closed under all $\mathcal{L}(\emptyset)$-definable continuous functions $f\colon \mathcal{R} \to \mathcal{R}$. If $\mathcal{O}$ is a proper subring of T, it also is a valuation ring, and $(\mathcal{R}, \mathcal{O})$ is an ordered valued field (see [DL95]).

Let $\Gamma = \mathcal{R}^\times/\mathcal{O}^\times$ be the value group and we let $v\colon \mathcal{R}^\times \to \Gamma$ denote the surjective valuation map.

A **monomial group** $\mathfrak{M}$ is the image of a section s of the valuation v. We ask that

- $\mathfrak{M}$ is closed under all power functions, or that
- $\mathfrak{M}^{\succ} := \{\mathfrak{m} \in \mathfrak{M} : \mathfrak{m} > 1\}$ is closed under exp, respectively.

We will be working in mutually interpretable expansions of $\mathcal{L}$:

- $\mathcal{L}_{\Gamma,\boldsymbol{k},s}$, the three sorted valued fields expansion with the section s
- $\mathcal{L}_{\mathfrak{M}}$, the expansion by a predicate for $\mathfrak{M}$

In each case denote the expansion of T by $T_{\mathfrak{M}}$.

The Power Bounded Case

If T is power bounded then the following holds:

- Suppose T has quantifier elimination and a universal axiomatization. Then $T_{\mathfrak{M}}$ has quantifier elimination in $\mathcal{L}_{\Gamma,\boldsymbol{k},s}$.
- The theory $T_{\mathfrak{M}}$ is complete.
- If T is model complete, then $T_{\mathfrak{M}}$ is also model complete in the language $\mathcal{L}_{\mathfrak{M}}$.
- If $\mathcal{L}$ is finite and T decidable, $T_{\mathfrak{M}}$ is decidable.
- Γ is purely stably embedded as an ordered Λ-vector space and orthogonal to the residue field, which is purely stably embedded as a model of T.
- $T_{\mathfrak{M}}$ is distal, therefore NIP, but not strong.
- For $A \subseteq \mathcal{R}$, every $\mathcal{L}_{\mathfrak{M}}(A)$-definable subset of $\mathcal{R}$ is the union of an $\mathcal{L}_{\mathfrak{M}}(A)$-definable open set and finitely many $\mathcal{L}_{\mathfrak{M}}(A)$-definable discrete sets.

Example (Puiseux series)

The field of Puiseux series

$$\mathbb{R}((t^{1/\infty})) := \left\{ \sum_{i \geqslant i_0} c_i t^{i/n} : i_0 \in \mathbb{Z} \text{ and } n \in \mathbb{N} \right\}$$

is a model of T_{an} by interpreting analytic $f : [0,1]^n \to \mathcal{R}$ via their Taylor series expansions.

- T_{an} is power bounded.
- The convex hull of $\mathbb{R}$ (all series with only non-negative exponents of t) is a T_{an}-convex ring.
- The subgroup $t^{\mathbb{Q}} = \{t^q : q \in \mathbb{Q}\} \subseteq \mathbb{R}((t^{1/\infty}))^{>}$ is a monomial group.

The Exponential Case

If T defines an exponential function, then is a definable set $A \subseteq \mathcal{R} \models T_{\mathfrak{M}}$ with $\mathbb{N} \subseteq A$ such that if $a \in A \setminus \mathbb{N}$, then $a > \mathbb{N}$.
Consequently, $\mathbb{N}$ is externally definable in any model of $T_{\mathfrak{M}}$, and if T has an archimedean model, then $\mathbb{N}$ is definable in some model of $T_{\mathfrak{M}}$.

Now let T define an exponential function exp.

Fact

The additive group of $\mathcal{R}$ admits the direct sum decomposition $\mathcal{R} = \mathcal{O} \oplus \log(\mathfrak{M})$.

Inspired by Camacho's work on Hahn fields in [Cam18], let $a \in \mathcal{R}$ and $\mathfrak{m} \in \mathfrak{M}$. By the Fact above, there is a unique $b \in \mathfrak{m}\log(\mathfrak{M})$ with $a - b \in \mathfrak{m}\mathcal{O}$.
We define $a|_{\mathfrak{m}} := b$, so $(a, \mathfrak{m}) \mapsto a|_{\mathfrak{m}}$ is an $\mathcal{L}_{\mathfrak{M}}(\emptyset)$-definable function. We also define

$$\operatorname{supp}(a) := \{\mathfrak{m} \in \mathfrak{M} : v(a - a|_{\mathfrak{m}}) = v(\mathfrak{m})\},$$

so $\operatorname{supp}(a)$ is an $\mathcal{L}_{\mathfrak{M}}(a)$-definable subset of $\mathfrak{M}$.

Example (Transseries)

Let $\mathbb{T}$, the log-exp transseries, be a model of $T_{\text{an,exp}}$ extended by the convex hull of $\mathbb{R}$ as valuation ring and the transmonomials as monomial group $\mathfrak{M}$. Then for $a \in \mathbb{T}$:

- $a|_{\mathfrak{m}}$ is exactly the truncation of a at a transmonomial $\mathfrak{m}$,
- and $\operatorname{supp}(a)$ is exactly the support of a.

If $\mathfrak{m}$ is an infinitesimal transmonomial in $\mathbb{T}$, then

$$\operatorname{supp}\left(\frac{1}{1-\mathfrak{m}}\right) = \{\mathfrak{m}^n : n \in \mathbb{N}\}.$$

Thus, in $\mathbb{T}$ with a monomial group $\mathfrak{M}$ we can define

$$\mathbb{N} = \left\{ \frac{\log \mathfrak{n}}{\log \mathfrak{m}} : \mathfrak{n} \in \operatorname{supp}\left(\frac{1}{1-\mathfrak{m}}\right) \right\}.$$

Proposition

Let $\mathfrak{m} \in \mathfrak{M}$ with $\mathfrak{m} < 1$. Then

$$\mathfrak{m}^n \in \operatorname{supp}\left(\frac{1}{1-\mathfrak{m}}\right) \textit{ for all } n \in \mathbb{N},$$

and if $\mathfrak{n} \in \operatorname{supp}(1/(1-\mathfrak{m}))$, then either $\mathfrak{n} = \mathfrak{m}^n$ for some n or $\mathfrak{n} < \mathfrak{m}^n$ for all n.

Idea: Using the geometric series, we can write

$$\frac{1}{(1-\mathfrak{m})} = 1 + \mathfrak{m} + \mathfrak{m}^2 + \cdots + \varepsilon,$$

where $\varepsilon < \mathfrak{m}^n$ for all $n \in \mathbb{N}$.

Proof of the Exponential Case

Fix $\mathfrak{m} \in \mathfrak{M}$ with $\mathfrak{m} < 1$ and let A be the definable set

$$A := \left\{ a \in \mathcal{R} : \exp(a \log \mathfrak{m}) \in \operatorname{supp}\left(\frac{1}{1-\mathfrak{m}}\right) \right\}.$$

Then $a \in A$ if and only if $a \in \mathbb{N}$ or $a > \mathbb{N}$, by the Proposition. □

References

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