Lexicographic Exponentiation of Chains *

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In memory of Felix Hausdorff.

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Abstract

The lexicographic power Δ^Γ of chains Δ and Γ is, roughly, the Cartesian power $\prod_{\gamma\in\Gamma}\Delta$, totally ordered lexicographically from the left. Here the focus is on certain powers in which either $\Delta=\mathbb{R}$ or $\Gamma=\mathbb{R}$, with emphasis on when two such powers are isomorphic and on when Δ^Γ is 2-homogeneous. The main results are:

1) For a countably infinite ordinal α, $\mathbb{R}^{\alpha^*+\alpha}\simeq\mathbb{R}^\alpha$.

2) $\mathbb{R}^\mathbb{R}\not\simeq\mathbb{R}^\mathbb{Q}$.

3) For Δ a countable ordinal ≥ 2, $\Delta^\mathbb{R}$, with its smallest element deleted, is 2-homogeneous.

1 Introduction

The study of lexicographic powers of chains (totally ordered sets) goes back to Hausdorff [H1] [1] and [H2]. Let $\Gamma\neq\emptyset$ be an index chain, and Δ a chain (the **base chain**) with distinguished element 0 (the **base point**). The **support** of a sequence $s=(\delta_\gamma)_{\gamma\in\Gamma}\in\prod_{\gamma\in\Gamma}\Delta$ is support $(s):=\{\gamma\in\Gamma\,;\,s(\gamma)=\delta_\gamma\neq 0\}$. The **lexicographic power** (computed in base 0) is the chain obtained as follows. We consider the following subset of $\prod_{\gamma\in\Gamma}\Delta$:

$$\Delta^\Gamma:=\{s:\Gamma\to\Delta\,;\,\text{support}\,(s)\text{ is wellordered}\}=\{s\in\prod_{\gamma\in\Gamma}\Delta\,;\,\text{support}\,(s)\text{ is wellordered}\},$$

which we order lexicographically from the left (also known as "order by first differences"). That is, for distinct $s=(\delta_\gamma)_{\gamma\in\Gamma}$ and $s'=(\delta'_\gamma)_{\gamma\in\Gamma}\in\Delta^\Gamma$, we let γ_0 be the smallest $\gamma\in\Gamma$ for which $s(\gamma)\neq s'(\gamma)$ (γ_0 exists since support $(s)\cup$ support (s') is wellordered), and we set $s<s'$ iff $s(\gamma_0)<s'(\gamma_0)$. In the sequel, when s and t are distinct elements of a lexicographic power Δ^Γ we will denote by $\mathbf{dif}(s,t)$ the smallest $\gamma\in\Gamma$ for which $s(\gamma)\neq t(\gamma)$.

Although Δ^Γ depends on the base point 0 (cf. Remark 2.8), our notation in this paper will not reflect this dependence. Recall that a chain Δ is said to be **homogeneous**

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[1] A translation into English of this paper appears as an appendix in [G].

(or **transitive**) if given a, b elements of Δ, there exists an automorphism σ of Δ such that $\sigma(a) = b$ (for example, the underlying chain of a totally ordered group A is always homogeneous: in the additive notation, given a and b as above define $\sigma(x) = x + (b - a)$). If Δ is homogeneous, then the lexicographic powers Δ^Γ are all isomorphic for any choice of the base point 0 of Δ (cf. Remark 2.8). Therefore when the base Δ is $\mathbb{R}$ we shall always assume that the base point 0 is the usual real 0. Also, when Δ is an ordinal, we assume unless specified otherwise that the base point is its least element 0 (but here not all base points need give isomorphic lexicographic powers; cf. the discussion in Section 2).

Our notation for lexicographic powers differs slightly from Hausdorff's: our Δ^Γ is written by Hausdorff as Δ^{Γ^*}. (Γ^* denotes the dual of Γ, that is Γ with its order reversed.) In the special case when α and β are ordinals, our α^{β^*} is (isomorphic to) the *ordinal* α^β [H1]. (We suspect that Hausdorff's notation for lexicographic powers was chosen precisely to be consistent with Cantor's notation for ordinal exponentiation.)

In [H2] Hausdorff's major interest in lexicographic powers is in their 2-homogeneity: a chain A (containing more than 2 elements) is said to be **2-homogeneous** (or **2-transitive**) if given a_1, a_2, b_1, b_2 elements of A such that $a_1 < b_1$ and $a_2 < b_2$, there exists an automorphism σ of A such that $\sigma(a_1) = a_2$ and $\sigma(b_1) = b_2$. If A is 2-homogeneous, then all open intervals of A are isomorphic; and conversely, provided A has no endpoints. Also if A is 2-homogeneous then it is n-homogeneous for all natural numbers $n \geq 2$ (defined analogously). Also if A is 2-homogeneous then so is A^*.

Example 1.1 The underlying chain of a totally ordered field F is always 2-homogeneous. In fact given a_1, a_2, b_1, b_2 as above define $\sigma(a) = (a - a_1)\frac{(b_2 - a_2)}{(b_1 - a_1)} + a_2$.

Let α be a nonzero ordinal. There are uniquely determined ordinals $\alpha_1 \geq \ldots \geq \alpha_n$ such that $\alpha = \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}$. This is called the **Cantor normal form** of α and α is called **additive principal** if $n = 1$ in its Cantor normal form. That is, α is additive principal if it is an ordinal power of ω, or equivalently, if α is isomorphic to any of its nonempty final segments [W; Proposition 2.8].

We mention now two results of Hausdorff's [H2, pp. 173–178] that relate to the material of our paper. We shall not make any use of them other than one in Corollary 3.6, which is not itself used for anything else in this paper.

Theorem 1.2 (Hausdorff) *Let κ be an infinite regular cardinal (or more generally, an infinite additive principal ordinal), and let Δ be a chain with base point* 0*. Suppose that Δ has no endpoints. Then Δ^κ is 2-homogeneous.*

Hausdorff's argument proves Theorem 1.2 for κ regular, though he does not explicitly state this theorem; and his argument can be adapted to deal with the general case. For the case where $\Delta = \mathbb{R}$, cf. [W; Proposition 2.14]:

Corollary 1.3 (Hausdorff, Warton) *Let α be an ordinal. If α is additive principal then $\mathbb{R}^\alpha$ is 2-homogeneous.*

The converse of this corollary is also true, as we shall prove in Section 4, Theorem 4.5.

The second result of Hausdorff is not about the full lexicographic power Δ^κ, but rather about its λ–**restricted** subchains: for an ordinal $\lambda \leq \kappa$ we let $(\Delta^\kappa)_\lambda$ denote the subchain consisting of those sequences s for which the order type of support (s) is $< \lambda$.

Theorem 1.4 (Hausdorff) *Suppose the hypotheses of Theorem 1.2 hold, except that now Δ is permitted to have endpoints provided its base point 0 is not an endpoint. Let λ be an infinite regular cardinal (or more generally, an infinite additive principal ordinal) such that $\lambda \leq \kappa$. Then $(\Delta^\kappa)_\lambda$ is 2-homogeneous.*

This time the result is explicitly stated by Hausdorff for the case κ and λ are both regular, and again can be adapted to deal with the general case.

In [K1], the second author studied lexicographic powers of the form $\mathbb{R}^\Gamma$. The main result of [K1] is the following:

Theorem 1.5 *Let α be an ordinal, and J a chain in which the chain $\mathbb{R}$ does not embed. Assume that φ is an embedding of $\mathbb{R}^\alpha$ in $\mathbb{R}^J$. Then there is $s \in \operatorname{im}\varphi$ such that the order type of* support (s) *is $\geq \alpha$.*

Corollary 1.6 *Let α be an ordinal, and J a chain in which the chain $\mathbb{R}$ does not embed. Assume that $\mathbb{R}^\alpha$ embeds in $\mathbb{R}^J$. Then α embeds in J. In particular, if α and β are distinct ordinals, then $\mathbb{R}^\alpha \not\simeq \mathbb{R}^\beta$.*

In this paper, we study in more detail the relation between the lexicographic power and its exponent. We focus on powers in which either $\Delta = \mathbb{R}$ or $\Gamma = \mathbb{R}$, with emphasis on when two such powers are isomorphic and on when Δ^Γ is 2-homogeneous. For example Corollary 1.6 says that

$$\text{if } \mathbb{R}^\Gamma \simeq \mathbb{R}^{\Gamma'} \text{then } \Gamma \simeq \Gamma'$$

holds when Γ and Γ' are ordinals. Does it hold for arbitrary chains Γ and Γ'? Addressing this question, we prove in Section 3 that:

Theorem A (3.4) *If α is any countably infinite ordinal, then $\mathbb{R}^{\alpha^*+\alpha} \simeq \mathbb{R}^\alpha$.*

On the other hand in Section 5 we find that

Theorem B (5.2) $\mathbb{R}^\mathbb{R} \not\simeq \mathbb{R}^\mathbb{Q}$.

The last main theorem, in Section 6, is

Theorem C (6.1) *Let Δ be a countable ordinal ≥ 2, with its least element 0 as base point. Then $\Delta^\mathbb{R}$ (with its minimum element deleted) is 2-homogeneous.*

We mention that Sections 4, 5 and 6 are independent of each other and can be read in any order.

An early application of lexicographic powers (even preceding Hausdorff) was the theorem of H. Hahn (cf. [F]) that every abelian totally ordered group embeds in a lexicographic power $\mathbb{R}^\Gamma$ endowed with the obvious additive group structure. He also investigated *formal power series* $F((G))$ where F is an ordered field and G an ordered group. The underlying chain of $F((G))$ is just the lexicographic power F^G (see [F]). More recently, lexicographic powers have found interesting applications to the study of convex congruences of $\mathrm{Aut}(\mathbb{R}^\alpha)$,

α an ordinal (cf. [W]), and to ordered exponential fields (cf. [K2], [K–K–S1], [K–K–S2] and [K–S]). We hope to investigate further their properties in future work.

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2 Definitions, arithmetic rules, and coterminalities

Let us introduce some notation and terminology. Let Γ and Γ' be chains. The **sum** $\Gamma + \Gamma'$ is the chain formed by concatenation, with $\Gamma < \Gamma'$. More generally, if $\{\Gamma_i; i \in I\}$ is a collection of chains indexed by a chain I, we define the sum $\sum_{i\in I} \Gamma_i$ analogously. Note that our definition coincides with ordinal addition in case Γ and Γ' are ordinals. We denote by $\Gamma \vec{\amalg} \Gamma'$ the **lexicographic product** of Γ and Γ'. That is, $\Gamma \vec{\amalg} \Gamma'$ is the chain obtained by ordering the Cartesian product $\Gamma \times \Gamma'$ lexicographically from the left. If α and β are ordinals then $\alpha \vec{\amalg} \beta$ is (isomorphic to) the ordinal product $\beta\alpha$.

To provide some context, we mention briefly the more general notion of **lexicographic product of a family of chains** $\{\Delta_\gamma \,;\, \gamma \in \Gamma\}$ with index chain Γ and for each $\gamma \in \Gamma$, a base point $0_\gamma \in \Delta_\gamma$. The definition is analogous to that for lexicographic powers. The lexicographic product (or Hahn product) is

$$\mathop{\mathbf{H}}_{\gamma\in\Gamma} \Delta_\gamma := \{s \in \prod_{\gamma\in\Gamma} \Delta_\gamma \,;\, \text{support}\,(s) \text{ is wellordered}\},$$

totally ordered lexicographically from the left; where support $(s) := \{\gamma \in \Gamma \,;\, s(\gamma) \neq 0_\gamma\}$.

Lexicographic exponentiation of chains: When all Δ_γ's are the same chain Δ, and all base points 0_γ are the same element $0 \in \Delta$, $\mathbf{H}_{\gamma\in\Gamma} \Delta_\gamma$ is the lexicographic power Δ^Γ discussed in the Introduction.

Proposition 2.1 *Δ^Γ has a minimum (respectively, maximum) element s_0 iff Δ has a minimum (respectively, maximum) element δ_0 and either the base point $0 = \delta_0$, or Γ is well ordered. When these conditions obtain, then s_0 is the constant function $s_0(\gamma) = \delta_0$ for all $\gamma \in \Gamma$.*

Proof: The only possible candidate for a minimal (maximal) element of Δ^Γ is the above s_0. The rest follows easily. □

We shall denote by $\mathbf{0}$ the sequence with empty support in Δ^Γ, that is, the constant sequence $s \in \Delta^\Gamma$ defined by $s(\gamma) = 0$ for all $\gamma \in \Gamma$. We note from the above proposition that $\mathbf{0}$ is the minimum (respectively, maximum) element of Δ^Γ if and only if 0 is the minimum (respectively, maximum) element of Δ.

Anti-lexicographic exponentiation of chains: The **anti-lexicographic power** ${}^\Gamma\Delta$ is the set

$$\{s : \Gamma \to \Delta \,;\, \text{support}(s) \text{ is anti-wellordered in } \Gamma\},$$

ordered anti-lexicographically, i.e., from the right (also known as "ordered by last differences").

The proofs of the following three results are straightforward:

Proposition 2.2 *Let Γ be a chain, and Δ a chain with a base point 0. Then the anti-lexicographic power ${}^{\Gamma}\Delta$ coincides with the lexicographic power Δ^{Γ^*}.*

Although Proposition 2.2 provides a simple way of "translating" results from the lexicographic notation to the anti-lexicographic notation, one has to be very careful in doing so. For example, if two lexicographic powers are isomorphic, the corresponding anti-lexicographic powers need not be (see Example 4.4).

Proposition 2.3 *Let Γ be a chain and Δ a chain with base point 0. Then $(\Delta^{\Gamma})^*$ coincides with the lexicographic power $(\Delta^*)^{\Gamma}$.*

Lemma 2.4 *Let Γ be a chain and Δ a chain with base point 0. Let Δ_1 be a chain and suppose that there is an isomorphism $\phi : \Delta \to \Delta_1$. Then ϕ lifts to an isomorphism*

$$\hat{\phi} : \Delta^{\Gamma} \to \Delta_1^{\Gamma}$$

where the base point of Δ_1 is taken to be $\phi(0)$.

Proof: For $s \in \Delta^{\Gamma}$ and $\gamma \in \Gamma$, define $(\hat{\phi}(s))(\gamma) = \phi((s(\gamma))$. $\square$

A chain A is **symmetric** if there is an order-reversing bijection (an anti-isomorphism) $\phi : A \to A$ (or equivalently, an isomorphism $\phi : A \to A^*$).

Corollary 2.5 *Let Γ and Δ be chains with $0 \in \Delta$ a base point. Suppose that Δ is symmetric, with an anti-isomorphism $\phi : \Delta \to \Delta$. Then $\Delta^{\Gamma} \simeq (\Delta^*)^{\Gamma} = (\Delta^{\Gamma})^*$, where the two last lexicographic powers are computed in base $\phi(0)$. In particular, if $\phi(0) = 0$ then Δ^{Γ} is symmetric.*

Corollary 2.6 *Let Γ and Δ be chains, with $\Delta = n = \{0, \cdots, n-1\}$, $n > 1$ finite. Then for every $i \in \Delta$ we have $\Delta^{\Gamma} \simeq (\Delta^{\Gamma})^*$, where the first lexicographic power is computed in base i, and the second lexicographic power is computed in base $n - (i+1)$. In particular, if n is odd then Δ^{Γ}, computed in base $\frac{n-1}{2}$ (the midpoint of n) is symmetric.*

From Corollary 2.5, we see that whenever Δ^{Γ} is independent (up to isomorphism) of the choice of the base point $0 \in \Delta$, then symmetry of Δ implies symmetry of Δ^{Γ}. However, as mentioned in the Introduction, lexicographic powers depend in general on the choice of the base point (see example at the end of Remark 2.7). The following is a brief analysis of this issue (we return to symmetry in Corollary 2.9 after this discussion).

Remark 2.7 Relation to ordinal exponentiation:
As mentioned in the Introduction, when α and β are ordinals, our lexicographic power α^{β^*} is the ordinal α^{β} (or the anti-lexicographic power ${}^{\beta}\alpha$ is the ordinal α^{β}).
We remind the reader that the chosen base point here is the least element $0 \in \alpha$. For example if α is the ordinal $2 = \{0, 1\}$, then the lexicographic power 2^{β^*}, when computed in base $1 \in \{0, 1\}$ instead of 0, is the *reverse* of the ordinal 2^{β}. Indeed by Corollary 2.5, applied with the anti-automorphism which switches 0 and 1, we see that for any chain Γ, the lexicographic power 2^{Γ} computed in base $1 \in \{0, 1\}$ is isomorphic to the reverse of the lexicographic power 2^{Γ} computed in base 0.

Remark 2.8 Dependence on the chosen zero in lexicographic exponentiation of chains:

In [H1] Hausdorff introduces lexicographic products as follows. Given $\{\Delta_\gamma\ ;\ \gamma \in \Gamma\}$ with index chain Γ, define a partial order on the Cartesian product $\prod_{\gamma\in\Gamma}\Delta_\gamma$ by comparing two sequences s and t lexicographically from the left just in case support $(s,t) := \{\gamma \in \Gamma; s(\gamma) \neq t(\gamma)\}$ has a least element γ_0, and then defining $s < t$ iff $s(\gamma_0) < t(\gamma_0)$. Now define an equivalence relation on $\prod_{\gamma\in\Gamma}\Delta_\gamma$: $s \sim t$ if support (s,t) is wellordered. The equivalence classes are maximal chains in this partial order. Let $[s]$ denote the equivalence class of $s \in \prod_{\gamma\in\Gamma}\Delta_\gamma$. Then each $[s]$ is a lexicographic product defined by s, that is, with base points $0_\gamma = s(\gamma) \in \Delta_\gamma$. So if $t \sim s$ then the lexicographic product with base points $0_\gamma = t(\gamma)$ coincides with the lexicographic product defined by s, and conversely.

If Γ is wellordered then there is a unique equivalence class, and the lexicographic product of $\{\Delta_\gamma\ ;\ \gamma \in \Gamma\}$ with index chain Γ is uniquely determined (independent of the chosen base points). It is just $\prod_{\gamma\in\Gamma}\Delta_\gamma$ totally ordered lexicographically.

Note that s and t may still define *isomorphic* lexicographic products even if $t \not\sim s$. This is the case for example, as noted in the Introduction, if each of the Δ_γ 's is a homogeneous chain: generalizing the proof of Lemma 2.4, for each $\gamma \in \Gamma$ fix an automorphism π_γ of Δ_γ satisfying $\pi_\gamma(s(\gamma)) = t(\gamma)$. Then the π_γ's induce the required isomorphism in the obvious way. Moreover, this induced isomorphism maps base sequence to base sequence.

Corollary 2.9 *Assume that Δ is symmetric. Then Δ^Γ is symmetric (for any choice of the base point 0) if either Γ is wellordered or Δ is homogeneous.*

In general, our understanding of symmetry of lexicographic powers Δ^Γ is limited to the above results. If $\Delta \neq 1$, Γ is not wellordered and Δ^Γ has a minimum (dually, a maximum), then by Proposition 2.1 Δ^Γ has no maximum (dually, no minimum) and of course Δ^Γ is not symmetric; but what if the endpoint is deleted? See the problems at the end of Section 6.

Remark 2.10 To what extent can one hope to say that a chain isomorphism $\phi : \Delta_1^{\Gamma_1} \simeq \Delta_2^{\Gamma_2}$ between lexicographic powers must also preserve the $\mathbf{0}$'s, that is, $\phi(\mathbf{0}_1) = \mathbf{0}_2$?

(1) Assume that Γ_1 and Γ_2 are *not* wellordered. (If Γ_i is wellordered then the base point 0_i is irrelevant.) If one (and thus the other) of $\Delta_1^{\Gamma_1}, \Delta_2^{\Gamma_2}$ has a minimum (maximum), then $\mathbf{0}_1$ and $\mathbf{0}_2$ *are* these minimums (maximums) by Proposition 2.1, so indeed $\phi(\mathbf{0}_1) = \mathbf{0}_2$.

(2) Suppose that either Δ_i, or more generally, some $\Delta_i^{\Gamma_i}$ is homogeneous (as will be the case in almost all the lexicographic powers considered in this paper which do not meet the conditions of (1)). Then ϕ can be followed or preceded by an automorphism of $\Delta_i^{\Gamma_i}$ to obtain an isomorphism $\phi\prime : \Delta_1^{\Gamma_1} \simeq \Delta_2^{\Gamma_2}$ for which $\phi\prime(\mathbf{0}_1) = \mathbf{0}_2$.

We now gather some well known facts, most of them elementary enough that we can omit the proofs.

Lemma 2.11 *Let Γ and Γ' be any chains. Then $\Gamma \vec{\amalg}\, \Gamma' \simeq \sum_{\gamma\in\Gamma}\Gamma'$.*

Note that the operations $+$ and $\vec{\amalg}$ are both associative, but in general not commutative. Observe also that

Lemma 2.12 *Let* Γ, Γ' *and* Γ'' *be chains. Then* $(\Gamma+\Gamma')\,\vec{\amalg}\,\Gamma'' \simeq (\Gamma\,\vec{\amalg}\,\Gamma'') + (\Gamma'\,\vec{\amalg}\,\Gamma'')$.

Clearly $(\Gamma_1+\Gamma_2)^* \simeq \Gamma_2^* + \Gamma_1^*$ and $(\Gamma_1\,\vec{\amalg}\,\Gamma_2)^* \simeq \Gamma_1^*\,\vec{\amalg}\,\Gamma_2^*$.
The following result of [H1] will be very useful

Lemma 2.13 *The following rules hold for lexicographic powers:*

$$\Delta^{\Gamma+\Gamma'} \simeq \Delta^{\Gamma}\,\vec{\amalg}\,\Delta^{\Gamma'}.$$

More generally, if $\{\Gamma_i; i\in I\}$ *is a collection of chains indexed by a chain* I*, then*

$$\Delta^{\sum_{i\in I}\Gamma_i} \simeq \mathop{\mathbf{H}}_{i\in I}\Delta^{\Gamma_i},$$

where the base point of Δ^{Γ_i} *is* $\mathbf{0}$*, the sequence with empty support. In particular*

$$\Delta^{\Gamma\,\vec{\amalg}\,\Gamma'} \simeq (\Delta^{\Gamma'})^{\Gamma}.$$

Proof: The first assertion is clear. For the second, assume that $s\in\Delta^{\sum_{i\in I}\Gamma_i}$. For every $i\in I$, set $S_i = \text{support}\,(s)\cap\Gamma_i$. Each S_i is wellordered and $\{i; S_i\neq\emptyset\}$ is wellordered. Now define $s'\in\mathbf{H}_{i\in I}\Delta^{\Gamma_i}$ by setting $s'(i) = s\restriction\Gamma_i$. Clearly, $s\restriction\Gamma_i\in\Delta^{\Gamma_i}$ since support $(s\restriction\Gamma_i) = S_i$ is wellordered in Γ_i. So s' is well defined. Also support $(s') = \{i; S_i\neq\emptyset\}$ is wellordered in I. So $s'\in\mathbf{H}_{i\in I}\Delta^{\Gamma_i}$. Clearly the map $s\mapsto s'$ is order preserving and onto. For the last assertion compute:

$$\Delta^{\Gamma\,\vec{\amalg}\,\Gamma'} \simeq \Delta^{\sum_{\gamma\in\Gamma}\Gamma'} \simeq \mathop{\mathbf{H}}_{\gamma\in\Gamma}\Delta^{\Gamma'} \simeq (\Delta^{\Gamma'})^{\Gamma}.$$

$\square$

Recall that for ordinals α and β, the ordinal product $\beta\alpha$ is (isomorphic to) $\alpha\,\vec{\amalg}\,\beta$. So by Lemma 2.13 we get that $\Delta^{\beta\alpha}\simeq(\Delta^{\beta})^{\alpha}$.

Corollary 2.14 *The following implications hold for lexicographic powers:*
(1) $\Delta^{\Gamma_1}\simeq\Delta^{\Gamma_2}$ *and* $\Delta^{\Gamma_1'}\simeq\Delta^{\Gamma_2'} \Rightarrow \Delta^{\Gamma_1+\Gamma_1'}\simeq\Delta^{\Gamma_2+\Gamma_2'}$.
(2) $\Delta^{\Gamma_1}\simeq\Delta^{\Gamma_2} \Rightarrow \Delta^{\Gamma\,\vec{\amalg}\,\Gamma_1}\simeq\Delta^{\Gamma\,\vec{\amalg}\,\Gamma_2}$.

We now gather a few useful observations concerning homogeneous and 2-homogeneous lexicographic powers.

Proposition 2.15 *A lexicographic power* Δ^{Γ} *is homogeneous if* Δ *is homogeneous (for any choice of a base point* $0\in\Delta$*).*

Proof: Fix $0\in\Delta$. Let s and $t\in\Delta^{\Gamma}$ (where Δ^{Γ} is computed in base 0). As in Remark 2.8 the automorphisms π_γ of Δ satisfying $\pi_\gamma(s(\gamma)) = t(\gamma)$ induce an isomorphism between the lexicographic product with base points $s(\gamma)$ and the lexicographic product with base points $t(\gamma)$. But since $s\sim\mathbf{0}\sim t$ the products coincide with Δ^{Γ} (with base point 0), so that the induced isomorphism is an automorphism mapping s to t. $\square$

The following proposition, also proved in [G], provides sufficient conditions for a lexicographic power to be 2-homogeneous. Note that these conditions are not necessary (see

Corollary 2.23). For each $\gamma \in \Gamma$, we define a pair of equivalence relations $\mathcal{C}^\gamma$ and $\mathcal{C}_\gamma$ on Δ^Γ by setting

$$s\mathcal{C}^\gamma t \text{ if and only if } s(\gamma') = t(\gamma') \text{ for all } \gamma' < \gamma,$$

and

$$s\mathcal{C}_\gamma t \text{ if and only if } s(\gamma') = t(\gamma') \text{ for all } \gamma' \leq \gamma.$$

The equivalence classes of an $s \in \Delta^\Gamma$ will be denoted by $s\mathcal{C}^\gamma$ and $s\mathcal{C}_\gamma$, respectively. It is clear that each $s\mathcal{C}^\gamma$ and each $s\mathcal{C}_\gamma$ is convex.

Proposition 2.16 *A lexicographic power Δ^Γ is 2-homogeneous if Γ is homogeneous and Δ is 2-homogeneous.*

Proof: Let $s_1 < t_1$ and $s_2 < t_2$ for some elements $s_i, t_i \in \Delta^\Gamma$. Let $\gamma_i = \mathrm{dif}(s_i, t_i)$, and choose an automorphism ϕ of Γ such that $\phi(\gamma_1) = \gamma_2$. Then ϕ induces an automorphism $\bar{\phi}$ of Δ^Γ such that for each $x \in \Delta^\Gamma$ and $\gamma \in \Gamma$, $(\bar{\phi}(x))(\gamma) = x(\phi^{-1}(\gamma))$. Then $\mathrm{dif}(\bar{\phi}(s_1), \bar{\phi}(t_1)) = \gamma_2$, so we may assume that $\gamma_1 = \gamma_2 = \gamma$, say. We have $s_1(\gamma) < t_1(\gamma)$ and $s_2(\gamma) < t_2(\gamma)$, and we choose an automorphism ψ of Δ such that $\psi(s_1(\gamma)) = s_2(\gamma)$ and $\psi(t_1(\gamma)) = t_2(\gamma)$. Then ψ induces an automorphism $\bar{\psi}$ of Δ^Γ such that for each $x \in \Delta^\Gamma$,

$$(\bar{\psi}(x))(\lambda) = \begin{cases} \psi(x(\lambda)) & \text{if } \lambda = \gamma\,, \\ x(\lambda) & \text{if not.} \end{cases}$$

Then $\bar{\psi}(s_1\mathcal{C}_\gamma) = s_2\mathcal{C}_\gamma$ and $\bar{\psi}(t_1\mathcal{C}_\gamma) = t_2\mathcal{C}_\gamma$, so we may assume that $s_1\mathcal{C}_\gamma = s_2\mathcal{C}_\gamma < t_1\mathcal{C}_\gamma = t_2\mathcal{C}_\gamma$. Clearly (or see Lemma 3.1) each $\mathcal{C}_\gamma$-class $x\mathcal{C}_\gamma \simeq \Delta^\Phi$, where $\Phi = \{\alpha \in \Gamma\,;\, \gamma < \alpha\}$, so that $x\mathcal{C}_\gamma$ is homogeneous (by Proposition 2.15). Therefore we can choose an automorphism σ of $s_i\mathcal{C}_\gamma$ such that $\sigma(s_1) = s_2$ and, independently, an automorphism τ of $t_i\mathcal{C}_\gamma$ such that $\tau(t_1) = t_2$. Finally, let ρ be the automorphism of Δ^Γ which agrees with σ on $s_i\mathcal{C}_\gamma$ and with τ on $t_i\mathcal{C}_\gamma$, and is the identity elsewhere. Then ρ satisfies $\rho(s_1) = s_2$ and $\rho(t_1) = t_2$. $\square$

A subset C of a chain A is **cofinal** in A if for every $a \in A$, there is a $c \in C$ such that $c \geq a$ (**coinitial** is defined dually). The **cofinality** of a chain A is the least cardinal that embeds cofinally in A (the **coinitiality** is defined dually). The cofinality (coinitiality) is a regular cardinal (if A has a last element, the cofinality of A is 1, and dually for coinitiality), and it is an isomorphism invariant. If C is cofinal in A, then the cofinality of C equals that of A (dually for C coinitial). We say that a chain A is C_{00} or that A has **countable coterminalities** if both the cofinality and the coinitiality of A are equal to $\aleph_0$. Note that this is equivalent to the assertion that there is a coterminal (both coinitial and cofinal) subset of A isomorphic to $\mathbb{Z}$. We say that a point $\alpha \in A$ has **countable left character** if $\{\alpha' \in A\,;\, \alpha' < \alpha\}$ has countable cofinality, and similarly for right character. The following remark is useful:

Remark 2.17 Let Γ be a chain, and Δ a chain with base point $0 \in \Delta$. Assume that 0 is not an endpoint of Δ. Pick δ', $\delta \in \Delta$ such that $\delta' < 0 < \delta$. For each $\gamma \in \Gamma$, define s'_γ and $s_\gamma \in \Delta^\Gamma$ by setting:

$$s'_\gamma(\gamma') = \begin{cases} \delta' & \text{if } \gamma' = \gamma \\ 0 & \text{otherwise.} \end{cases} \qquad \text{and} \qquad s_\gamma(\gamma') = \begin{cases} \delta & \text{if } \gamma' = \gamma \\ 0 & \text{otherwise.} \end{cases}$$

Note that the s'_γ and s_γ have support $\{\gamma\}$, so they belong to $\overrightarrow{\Pi}_\Gamma\Delta$, where $\overrightarrow{\Pi}_\Gamma\Delta$ denotes the subchain of Δ^Γ consisting of those sequences with *finite* support. The maps $\gamma \mapsto s'_\gamma$ and $\gamma \mapsto s_\gamma$ define embeddings φ and φ^* of the chains Γ, respectively Γ^*, in $\overrightarrow{\Pi}_\Gamma\Delta$. Moreover we have $\varphi(\Gamma) < \{\mathbf{0}\} < \varphi^*(\Gamma^*)$ in $\overrightarrow{\Pi}_\Gamma\Delta$.

Note that if 0 is an endpoint, the above embeddings do not necessarily exist; for example if β is an infinite ordinal with base point the least element 0, then 2^{β^*} is an ordinal in which β^* cannot embed.

Proposition 2.18 *Let Γ be a chain, and Δ a chain with base point $0 \in \Delta$. Assume that 0 is not an endpoint of Δ, and that Γ has no least element. Let $\kappa \geq \aleph_0$ be the coinitiality of Γ. Then the cofinality and coinitiality of the lexicographic power Δ^Γ are equal to κ.*

Proof: Choose s'_γ and s_γ as in Remark 2.17. By the assumption on Γ, it is easily verified that $\{s'_\gamma\,;\, \gamma \in \Gamma\}$ is coinitial in Δ^Γ and has coinitiality κ. Similarly, $\{s_\gamma\,;\, \gamma \in \Gamma\}$ is cofinal in Δ^Γ and has cofinality κ. □

Mostly, we shall consider the special case of Proposition 2.18, when Γ has countable coinitiality:

Proposition 2.19 *Let Γ be a chain, and Δ a chain with base point $0 \in \Delta$. Then the lexicographic power Δ^Γ is C_{00} if either Γ has a least element and Δ is C_{00}, or Γ has countable coinitiality and 0 is not an endpoint of Δ.*

Proof: The conclusion is clear if Γ has a least element, and follows from Proposition 2.18 if Γ has countable coinitiality. □

Proposition 2.20 *Assume that A_1 and A_2 are C_{00} chains. Assume that all open intervals (a, b) of A_1 are isomorphic to all those of A_2. Then $A_1 \simeq A_2$.*

Proof: Patching. □

Corollary 2.21 *Let A be a 2-homogeneous C_{00} chain. Then A is isomorphic to any of its convex C_{00} subsets.*

Proof: Let C be a convex C_{00} subset of A. By convexity, any interval of C is an interval of A. Since A is 2-homogeneous, the assertion now follows by Proposition 2.20. □

Note that the condition "countable coterminalities" is necessary in the last two results: Consider the following two "long rational lines": let (0,1) be the open rational interval, and let [0,1) be the half-open rational interval. Let A' be the lexicographic product $\omega_1 \overrightarrow{\Pi}\,(0,1)$ and $A = 1 + A'$. Let B be the lexicographic product $\omega_1 \overrightarrow{\Pi}\ [0,1)$. Then each open interval of A or of B is isomorphic to the rationals. Both A and B have a least element, and cofinality $\aleph_1$. It can be shown that these two chains are *not* isomorphic.

A special case of the following theorem appears in [G].

Theorem 2.22 *Let Γ be a homogeneous chain of countable coinitiality which is isomorphic to one (and hence each) of its open final intervals (γ, ∞). Let Δ be a 2-homogeneous C_{00} chain. Then $\Delta^\Gamma \simeq \Delta^{1+\Gamma} \simeq \Delta^{n+\Gamma}$ for any natural number n.*

Proof: By definition of 2-homogeneity, $|\Delta| > 2$ and Δ has no endpoints, so 0 is not an endpoint of Δ. Hence, by Proposition 2.19, Δ^Γ is C_{00}. By Proposition 2.16, Δ^Γ is 2-homogeneous. It follows from Corollary 2.21 that Δ^Γ is isomorphic to each of its convex C_{00} subsets. Pick any $\gamma \in \Gamma$. Then $0\mathcal{C}^\gamma$ is a convex subset of Δ^Γ, and because Δ is C_{00}, so is $0\mathcal{C}^\gamma$. Therefore, $\Delta^\Gamma \simeq 0\mathcal{C}^\gamma$. Also, $0\mathcal{C}^\gamma \simeq \Delta^{1+\Gamma}$. Finally, $\Delta^{1+\Gamma} \simeq \Delta^{n+\Gamma}$ by induction using Lemma 2.13. $\square$

Corollary 2.23 $\mathbb{R}^{\mathbb{Q}} \simeq \mathbb{R}^{1+\mathbb{Q}}$ *and* $\mathbb{R}^{\mathbb{R}} \simeq \mathbb{R}^{1+\mathbb{R}}$.

Theorem 2.24 $\mathbb{R}^{\mathbb{R}} \simeq \mathbb{R}^{\mathbb{R}+\mathbb{R}}$, *and these chains are 2-homogeneous.*

Proof: By Lemma 2.13, $\mathbb{R}^{\mathbb{R}+\mathbb{R}} \simeq \mathbb{R}^{\mathbb{R}} \vec{\amalg} \mathbb{R}^{\mathbb{R}}$. Moreover, $\mathbb{R}^{\mathbb{R}} \simeq \mathbb{R}^{(-\infty,0)}$ (since the exponents are isomorphic). Also $\mathbb{R}^{\mathbb{R}} \simeq \mathbb{R}^{1+(0,\infty)}$ by Corollary 2.23. Thus $\mathbb{R}^{\mathbb{R}+\mathbb{R}} \simeq \mathbb{R}^{(-\infty,0)} \vec{\amalg} \mathbb{R}^{1+(0,\infty)} \simeq \mathbb{R}^{\mathbb{R}}$, by Corollary 2.14. $\mathbb{R}^{\mathbb{R}}$ is 2-homogeneous by Proposition 2.16. $\square$

We mention that the above result remains true if $\mathbb{R}$ is replaced by any 2-homogeneous C_{00} chain with countable point characters.

3 The chains $\mathbb{R}^{\alpha^*+\alpha}$

In this section, we shall prove our first main result, Theorem A, that for a countably infinite ordinal α, $\mathbb{R}^{\alpha^*+\alpha} \simeq \mathbb{R}^\alpha$. For a lexicographic power $\Delta^{\Gamma_1+\Gamma_2}$, we shall sometimes speak of Δ^{Γ_2} as being a convex subset of $\Delta^{\Gamma_1+\Gamma_2}$. When we do so, we are identifying these two chains via the map π in the following lemma, the proof of which is obvious.

Lemma 3.1 *Let Γ be a chain and Δ a chain with base point 0, and let $\Phi \neq \emptyset$ be a final segment of Γ. Let $\pi : \Delta^\Phi \to \Delta^\Gamma$ be defined by $(\pi(s))(\gamma) = s(\gamma)$ if $\gamma \in \Phi$, and $(\pi(s))(\gamma) = 0$ otherwise. Then π is an isomorphism onto a convex subset of Δ^Γ, and $\pi(\mathbf{0}) = \mathbf{0}$.*

Note that the converse fails; a convex subset of Δ^Γ *need not* be isomorphic to a Δ^Φ. The following lemma, two theorems and corollary, remain valid if $\mathbb{R}$ is replaced by any 2-homogeneous C_{00} chain.

Lemma 3.2 $\mathbb{R}^{\omega^*+\omega} \simeq \mathbb{R}^\omega$, *and these chains are 2-homogeneous.*

Proof: $\mathbb{R}^\omega$ is convex in $\mathbb{R}^{\omega^*+\omega}$ by Lemma 3.1, and both are C_{00} chains by Proposition 2.19. By Proposition 2.16, $\mathbb{R}^{\omega^*+\omega}$ is a 2-homogeneous chain. So again Corollary 2.21 applies. $\square$

Theorem 3.3 *Let α be a countable ordinal. Then $\mathbb{R}^{\alpha^*+\omega} \simeq \mathbb{R}^\omega$ and these chains are 2-homogeneous.*

Proof: Note that $\mathbb{R}^\omega$ is 2-homogeneous by Lemma 3.2. We prove that $\mathbb{R}^{\alpha^*+\omega} \simeq \mathbb{R}^\omega$ by induction on α. For α finite the assertion is clear. Suppose that the assertion is true for α. Then $\mathbb{R}^{(\alpha+1)^*+\omega} \simeq \mathbb{R}^{1+\alpha^*+\omega} \simeq \mathbb{R}^{1+\omega} \simeq \mathbb{R}^\omega$, by the induction hypothesis and

Corollary 2.14 . Suppose now that α is a limit ordinal. It suffices to show that $\mathbb{R}^{\alpha^*+\omega}$ is 2-homogeneous. Indeed once this is established, we can argue as in Lemma 3.2 to prove the main assertion (observe that $\mathbb{R}^{\alpha^*+\omega}$ has countable coterminalities since α is countable). Let $s_1, s_2, s_3, s_4 \in \mathbb{R}^{\alpha^*+\omega}$ be given such that $s_1 < s_2$ and $s_3 < s_4$. There exists an ordinal $\beta < \alpha$ such that $s_1, s_2, s_3, s_4 \in \mathbb{R}^{\beta^*+\omega}$. By induction, $\mathbb{R}^{\beta^*+\omega}$ is 2-homogeneous, so there exists an automorphism of this chain mapping s_1 to s_3 and s_2 to s_4. Since $\mathbb{R}^{\beta^*+\omega}$ is convex in $\mathbb{R}^{\alpha^*+\omega}$ (by Lemma 3.1), any automorphism of $\mathbb{R}^{\beta^*+\omega}$ extends to an automorphism of $\mathbb{R}^{\alpha^*+\omega}$. So $\mathbb{R}^{\alpha^*+\omega}$ is 2-homogeneous as required. □

The following theorem implies Theorem A:

Theorem 3.4 *Let α and β be ordinals, with α countable and β infinite. Then $\mathbb{R}^{\alpha^*+\beta} \simeq \mathbb{R}^{\beta}$.*

Proof: Write $\beta = \omega + \gamma$ for some ordinal γ. Then $\mathbb{R}^{\alpha^*+\beta} = \mathbb{R}^{\alpha^*+\omega+\gamma} \simeq \mathbb{R}^{\omega+\gamma} = \mathbb{R}^{\beta}$ (by Theorem 3.3 and Corollary 2.14). □

In view of Theorem 3.4 and Corollary 1.3 we have

Corollary 3.5 *Let α be a countably infinite, additive principal ordinal. Then $\mathbb{R}^{\alpha^*+\alpha}$ is a 2-homogeneous chain.*

The converse of Corollary 3.5 is also true, as we shall prove later (see Corollary 4.6).

Corollary 3.6 $\mathbb{R}^{\omega_1^*+\omega_1}$ *is a 2-homogeneous chain.*

Proof: For each countable ordinal α, the set $\alpha^* + \omega_1$ is a final segment of $\omega_1^* + \omega_1$. If we let

$$\Delta_\alpha = \{s \in \mathbb{R}^{\omega_1^*+\omega_1} ; \operatorname{dif}(s, \mathbf{0}) \in \alpha^* + \omega_1\},$$

then $\Delta_\alpha \simeq \mathbb{R}^{\alpha^*+\omega_1}$ and for $\alpha < \beta < \omega_1$, $\Delta_\alpha \subseteq \Delta_\beta$. Also,

$$\mathbb{R}^{\omega_1^*+\omega_1} = \bigcup_{\alpha<\omega_1} \Delta_\alpha.$$

By the theorem each $\Delta_\alpha \simeq \mathbb{R}^{\omega_1}$, and so by Corollary 1.3 Δ_α is 2-homogeneous. Since any automorphism of the convex subset Δ_α can clearly be extended to an automorphism of $\mathbb{R}^{\omega_1^*+\omega_1}$, the corollary follows. □

Theorems A, 3.3, and 3.4 are not true without the assumption of countability:

Example 3.7 Let κ be an uncountable regular cardinal. Then $\mathbb{R}^{\kappa^*+\kappa}$ and $\mathbb{R}^{\kappa}$ are *not* isomorphic. Indeed $\mathbb{R}^{\kappa}$ is C_{00} by Proposition 2.19, whereas $\mathbb{R}^{\kappa^*+\kappa}$ has cofinality κ by Proposition 2.18.

Example 3.8 In Theorem 3.3, $\mathbb{R}^{\alpha}$ itself need not be 2-homogeneous, even for countable α: In $\mathbb{R}^{\omega+1}$, each element is contained in a convex copy of $\mathbb{R}$ (consisting of those elements agreeing with it except in the last place) which has neither an infimum nor a supremum in $\mathbb{R}^{\omega+1}$. Hence $\mathbb{R}^{\omega+1}$ is not 2-homogeneous.

In fact, stronger results hold: we can prove the converses to Corollary 1.3 and Corollary 3.5 (see Theorem 4.5 and Corollary 4.6).

4 Nonisomorphism of lexicographic powers.

Theorem 1.5 and Corollary 1.6 provide tools for establishing nonisomorphism of lexicographic powers $\mathbb{R}^\Gamma$ with base $\mathbb{R}$, as Corollary 4.3 will show. The next Lemma ([K2; Lemma 4.9]) is the analogue to Lemma 2.4:

Lemma 4.1 *Let Γ and Γ' be chains and Δ a chain with base point 0. Suppose that $\varphi : \Gamma \to \Gamma'$ is an embedding. Then φ lifts to an embedding*

$$\hat{\varphi} : \Delta^\Gamma \to \Delta^{\Gamma'}$$

such that $\hat{\varphi}$ takes the base (constant 0) function of Δ^Γ to the base function of $\Delta^{\Gamma'}$; namely, $(\hat{\varphi}(s))(\gamma') = 0$ if $\gamma' \notin \varphi(\Gamma)$, and $(\hat{\varphi}(s))(\varphi(\gamma)) = s(\gamma)$ otherwise. Moreover $\hat{\varphi}$ is onto if and only if φ is.

In her dissertation [W, Theorem 3.1], Pam Warton shows that Corollary 1.6 holds even if one drops the condition on J. However, the condition on J is necessary for the conclusion of Theorem 1.5 as the following example shows.

Example 4.2 In [K–K–S2] there was constructed a chain Γ such that $\mathbb{R}$ embeds in Γ, and $(\mathbb{R}^\Gamma)^{\leq \mathbf{0}} \simeq \Gamma$ (here $(\mathbb{R}^\Gamma)^{\leq \mathbf{0}}$ denotes the closed initial segment of $\mathbb{R}^\Gamma$ determined by $\mathbf{0}$). Let J be the chain obtained by deleting the last element of Γ. Then by Lemma 4.1 $(\mathbb{R}^J)^{<\mathbf{0}}$ embeds in $(\mathbb{R}^\Gamma)^{<\mathbf{0}} \simeq J$. Moreover, the map $s \mapsto -s$ from $(\mathbb{R}^\Gamma)^{<\mathbf{0}}$ to $(\mathbb{R}^\Gamma)^{>\mathbf{0}}$ defined by $(-s)(\gamma) = -(s(\gamma))$ is an order-reversing bijection. Therefore, $(\mathbb{R}^\Gamma)^{>\mathbf{0}} \simeq ((\mathbb{R}^\Gamma)^{<\mathbf{0}})^* \simeq J^*$. Now we have

$$\mathbb{R}^J = (\mathbb{R}^J)^{<\mathbf{0}} + \{\mathbf{0}\} + (\mathbb{R}^J)^{>\mathbf{0}} .$$

Thus $\mathbb{R}^J$ embeds in $J + \{\mathbf{0}\} + J^*$. By Remark 2.17, it follows that $\mathbb{R}^J$ embeds in $\vec{\amalg}_J\mathbb{R}$. But J contains an infinite wellordered subset α, and $\mathbb{R}^\alpha$ embeds in $\mathbb{R}^J$ (by Lemma 4.1), and hence in $\vec{\amalg}_J\mathbb{R}$. This violates the conclusion of Theorem 1.5 because the elements of $\vec{\amalg}_J\mathbb{R}$ have finite support.

We now apply Corollary 1.6 and its improvement in [W], along with Lemma 4.1. If Γ is a chain we define the following set of ordinals:

$$wo(\Gamma) = \{\alpha\,;\ \alpha \text{ is the order type of some wellordered subset of } \Gamma\}.$$

Corollary 4.3 *Let Γ_1 and Γ_2 be arbitrary chains. If $\mathbb{R}^{\Gamma_1}$ embeds in $\mathbb{R}^{\Gamma_2}$ then $wo(\Gamma_1) \subset wo(\Gamma_2)$. Consequently:*
(1) If $\mathbb{R}^{\mathbb{Q}}$ embeds in some $\mathbb{R}^\Gamma$, then all countable ordinals embed in Γ, and in particular Γ cannot be of the form $\beta^* + \delta$, with β and δ ordinals and δ countable.
(2) If $\mathbb{R}^\alpha$ embeds in $\mathbb{R}^{\mathbb{R}}$ for some ordinal α, then α must be countable.

Example 4.4 We observed in Lemma 3.2 that $\mathbb{R}^{\omega^*+\omega} \simeq \mathbb{R}^\omega$. However the corresponding anti-lexicographic powers are not isomorphic. If they were, then we would have by Proposition 2.2 that $\mathbb{R}^{\omega^*+\omega} \simeq \mathbb{R}^{\omega^*}$, so $\mathbb{R}^\omega \simeq \mathbb{R}^{\omega^*}$. But this is impossible by Corollary 4.3 since ω does not embed in ω^*.

We have the promised converse of Theorem 1.3 :

Theorem 4.5 *Let α be an ordinal such that $\mathbb{R}^\alpha$ is 2-homogeneous. Then α is additive principal.*

Proof: Assume that $\alpha \neq 1$. Let φ be a nonempty final segment of α. Both $\mathbb{R}^\alpha$ and $\mathbb{R}^\varphi$ are C_{00} by Proposition 2.19, and $\mathbb{R}^\varphi$ is convex by Lemma 3.1. So by Corollary 2.21 $\mathbb{R}^\alpha \simeq \mathbb{R}^\varphi$. So $\alpha \simeq \varphi$ by Corollary 1.6, this shows that α is self final as required. $\square$

In view of Theorem 3.4, we have the converse of Corollary 3.5:

Corollary 4.6 *Let α be a countably infinite ordinal. Assume that $\mathbb{R}^{\alpha^*+\alpha}$ is a 2-homogeneous chain. Then α is additive principal.*

5 $\mathbb{R}^\mathbb{R}$ is not isomorphic to $\mathbb{R}^\mathbb{Q}$

Let us first recall some definitions and facts. Let A be a totally ordered set. Let X, Y be subsets of A. We write $X < Y$ if $x < y$ for all $x \in X$ and $y \in Y$. A **Dedekind cut** in A is a pair (X, Y) of disjoint nonempty convex subsets of A whose union is A and $X < Y$. A Dedekind cut is a **gap** or a **hole** in A if X has no last element and Y has no first element, and is a **jump** if X has a last element and Y has a first element. A dense ordering has no jumps. A is said to be **Dedekind complete** if there are no gaps in A. For example, $\mathbb{R}$ is Dedekind complete. We denote by $\overline{A}$ the **Dedekind completion** of a chain A.

Certainly $\mathbb{R}^\mathbb{R}$ and $\mathbb{R}^\mathbb{Q}$ are 2-homogeneous (by 2.16) and C_{00} (by 2.19). However, the above techniques do not enable us to determine whether $\mathbb{R}^\mathbb{R} \simeq \mathbb{R}^\mathbb{Q}$. This section is devoted to the proof of the following theorem and corollary:

Theorem 5.1 *Let Δ and Γ be Dedekind complete ordered sets and let Δ have no endpoints. Let Δ' be any ordered set, and let Γ' be a countable ordered set which is dense in itself. Then $\Delta^\Gamma \not\simeq (\Delta')^{\Gamma'}$.*

Corollary 5.2 $\mathbb{R}^\mathbb{R} \not\simeq \mathbb{R}^\mathbb{Q}$.

Let $0' \in \Delta'$ be the chosen base point, and $\mathbf{0'} \in (\Delta')^{\Gamma'}$ the corresponding base function. The proof of the theorem begins by noting that the set $(\Delta')^{\Gamma'}$ has a natural partition into convex subsets, which we will call *cells*, where two elements g, h belong to the same cell iff $\mathrm{dif}(g, \mathbf{0'}) = \mathrm{dif}(h, \mathbf{0'})$ and either $g, h < \mathbf{0'}$ or $g, h > \mathbf{0'}$. The element $\mathbf{0'}$ constitutes a cell by itself. Then the collection of cells is a countably infinite ordered set which, with respect to its naturally induced order, is dense in itself. We will show that Δ^Γ has no such partition.

The **neighborhood** corresponding to a pair $(a, \gamma) \in \Delta^\Gamma \times \Gamma$ is

$$N_{a,\gamma} = a\mathcal{C}^\gamma = \{d \in \Delta^\Gamma; \mathrm{dif}(a, d) > \gamma\}.$$

The definition of the lexicographic power implies that if $N_{a_1,\gamma_1} \supseteq N_{a_2,\gamma_2} \supseteq \ldots$ is a tower of neighborhoods, then $\bigcap N_{a_i,\gamma_i} \neq \emptyset$.

Let $\mathfrak{C} = \{C_1, C_2, \ldots\}$ be a countably infinite convex partition of Δ^Γ such that in its natural order, $\mathfrak{C}$ is dense in itself. We will obtain a contradiction. Our procedure will be to define inductively a tower of neighborhoods $N_{a_1,\gamma_1} \supseteq N_{a_2,\gamma_2} \supseteq \ldots$ such that for each i, $N_{a_i,\gamma_i} \cap C_i = \emptyset$. Then any point of the intersection of the tower will be outside all the cells of $\mathfrak{C}$.

Since Δ and Γ are both Dedekind complete, whereas $\mathfrak{C}$ is isomorphic to an interval of $\mathbb{Q}$, we have:

Lemma 5.3 *There is no order-preserving map from an interval of Δ or of Γ onto any non-trivial interval of $\mathfrak{C}$.*

We will invoke Lemma 5.3 twice in the proof of Lemma 5.4 below, once for Δ and once for Γ. Lemma 5.4 will enable us to construct, inductively, a nested sequence of neighborhoods N_{a_i,γ_i} whose nonvoid intersection misses each of the cells in $\mathfrak{C}$. The first conclusion of the lemma allows us to eliminate, successively, each of the cells. The second insures that none of our neighborhoods is trapped entirely within one cell (which would otherwise make it impossible to eventually eliminate that cell).

Lemma 5.4 *Let Δ and Γ be Dedekind complete ordered sets and let Δ have no endpoints. Let $\mathfrak{C}$ be a countably infinite convex partition of Δ^Γ such that $\mathfrak{C}$ is dense in itself in the natural order. Let $C \in \mathfrak{C}$. Then there exists a neighborhood $N_{a,\gamma}$ and two distinct elements $A, B \in \mathfrak{C}$ such that*

(i) $N_{a,\gamma} \cap C = \emptyset$, while

(ii) $N_{a,\gamma} \cap A \neq \emptyset \neq N_{a,\gamma} \cap B$.

Proof of Theorem 5.1, using Lemma 5.4. Let $\mathfrak{C} = \{C_1, C_2, \ldots\}$ be a countably infinite convex partition of Δ^Γ which is dense in itself. We will arrive at a contradiction. By Lemma 5.4, there exists a neighborhood N_{a_1,γ_1} and distinct $A_1, B_1 \in \mathfrak{C}$ such that $C_1 \cap N_{a_1,\gamma_1} = \emptyset$ and $A_1 \cap N_{a_1,\gamma_1} \neq \emptyset \neq B_1 \cap N_{a_1,\gamma_1}$. By induction, suppose we have neighborhoods $N_{a_1,\gamma_1} \supseteq N_{a_2,\gamma_2} \supseteq \ldots \supseteq N_{a_n,\gamma_n}$ and $A_1, B_1, A_2, B_2, \ldots, A_n, B_n \in \mathfrak{C}$ such that for each i, $C_i \cap N_{a_i,\gamma_i} = \emptyset$ and $A_i \cap N_{a_i,\gamma_i} \neq \emptyset \neq B_i \cap N_{a_i,\gamma_i}$, and $A_i \neq B_i$. If $C_{n+1} \cap N_{a_n,\gamma_n} = \emptyset$, then letting $N_{a_{n+1},\gamma_{n+1}} = N_{a_n,\gamma_n}$, $A_{n+1} = A_n$, and $B_{n+1} = B_n$, we achieve the induction step. But if $C_{n+1} \cap N_{a_n,\gamma_n} \neq \emptyset$, then

$$C := C_{n+1} \cap N_{a_n,\gamma_n} \in \mathfrak{C}_n,$$

where

$$\mathfrak{C}_n := \{C_i \cap N_{a_n,\gamma_n}; C_i \in \mathfrak{C} \text{ and } C_i \cap N_{a_n,\gamma_n} \neq \emptyset\}.$$

Since $C_{n+1} \cap N_{a_n,\gamma_n} \neq \emptyset$, then $N_{a_n,\gamma_n} \neq \emptyset$, and so $(\gamma_n, \infty) \neq \emptyset$. Thus (see 3.1), we can identify the neighorhood N_{a_n,γ_n} with $\Delta^{(\gamma_n,\infty)}$, and with this identification, if N is a neighborhood of Δ^Γ such that N is properly contained in N_{a_n,γ_n}, then N is a neighborhood of $\Delta^{(\gamma_n,\infty)}$, and conversely. Clearly, (γ_n, ∞) is a Dedekind complete set. We show that $\mathfrak{C}_n$ is a partition of $\Delta^{(\gamma_n,\infty)}$ satisfying the hypotheses of Lemma 5.4 for $\Delta^{(\gamma_n,\infty)}$. First, $\mathfrak{C}_n$ is countably infinite since N_{a_n,γ_n} meets at least two different members of $\mathfrak{C}$ and $\mathfrak{C}$ is dense in itself. The other hypotheses are obviously satisfied. From Lemma 5.4, there exists a neighborhood $N_{a_{n+1},\gamma_{n+1}} \subseteq N_{a_n,\gamma_n}$ and distinct cells

$$A^*_{n+1} = A_{n+1} \cap N_{a_n,\gamma_n}, B^*_{n+1} = B_{n+1} \cap N_{a_n,\gamma_n} \in \mathfrak{C}_n$$

such that

$$\emptyset = C \cap N_{a_{n+1},\gamma_{n+1}} = C_{n+1} \cap N_{a_n,\gamma_n} \cap N_{a_{n+1},\gamma_{n+1}} = C_{n+1} \cap N_{a_{n+1},\gamma_{n+1}}$$

and

$$A^*_{n+1} \cap N_{a_{n+1},\gamma_{n+1}} \neq \emptyset \neq B^*_{n+1} \cap N_{a_{n+1},\gamma_{n+1}}.$$

Hence,

$$A_{n+1} \cap N_{a_{n+1},\gamma_{n+1}} \neq \emptyset \neq B_{n+1} \cap N_{a_{n+1},\gamma_{n+1}}$$

and $A_{n+1} \neq B_{n+1}$. The induction step is complete. Finally, since $\bigcap N_{a_i,\gamma_i} \neq \emptyset$, let $a \in \bigcap N_{a_i,\gamma_i}$. Then $a \notin \bigcup C_i$, contradicting the assumption that $\mathfrak{C}$ covers Δ^Γ.

Proof of Lemma 5.4. The dual Δ^* of Δ is also Dedekind complete and has no minimum or maximal elements. The identity map from Δ^Γ to $\Delta^{*\Gamma}$ reverses order, hence preserves convexity and the density of $\mathfrak{C}$, and preserves neighborhoods. Thus, we may assume that C is not the smallest member of $\mathfrak{C}$; otherwise, we deal with $\Delta^{*\Gamma}$ in which C is certainly not the smallest element, as $\mathfrak{C}$ is infinite.

We can now choose $C', C'' \in \mathfrak{C}$ with $C' < C'' < C$. Let $g \in C', m \in C''$, and let $\tau = \mathrm{dif}(g, m)$. Then using the fact that Δ has no minimum element,

$$[\tau, \infty) = \{\mathrm{dif}(a, m); g \leq a < m\} \subseteq \Gamma.$$

Suppose (falsely, as we shall see) that for each $\sigma \geq \tau \in \Gamma$, there is $A_\sigma \in \mathfrak{C}$ such that

$$P_\sigma := \{a \in \Delta^\Gamma; g \leq a < m \text{ and } \mathrm{dif}(a, m) = \sigma\} \subseteq A_\sigma.$$

We note that $P_\sigma \neq \emptyset$, and hence for each σ, A_σ is unique. Then we have a mapping $\phi : [\tau, \infty) \to \mathfrak{C}$ defined by $\phi(\sigma) = A_\sigma$. We will get a contradiction to Lemma 5.3 by showing that ϕ preserves order and has a non-trivial interval of $\mathfrak{C}$ in its image. The image $\phi([\tau, \infty))$ contains the non-trivial interval (C', C'') of $\mathfrak{C}$, for if $C' < T < C''$ and $T \in \mathfrak{C}$, let $t \in T$. Then $g < t < m$, so $\mathrm{dif}(t, m) \geq \tau$ and $T = \phi(\mathrm{dif}(t, m))$. Moreover, ϕ preserves order because if $\tau \leq \sigma_1 < \sigma_2 \in \Gamma$ and $\mathrm{dif}(s_i, m) = \sigma_i$ with $g \leq s_i < m$, then $s_1(\sigma_1) < m(\sigma_1) = s_2(\sigma_1)$, but for all $\sigma' < \sigma_1 \in \Gamma$, $s_1(\sigma') = m(\sigma') = s_2(\sigma')$. Hence, $s_1 < s_2$. And as $s_i \in P_{\sigma_i} \subseteq A_{\sigma_i}$, we have $A_{\sigma_1} \leq A_{\sigma_2}$. Thus, ϕ is an order-preserving map of $[\tau, \infty) \subseteq \Gamma$ whose image contains the non-trivial interval (C', C'') of $\mathfrak{C}$. We have a contradiction to Lemma 5.3. Therefore, our assumption about the existence of A_σ is wrong. Therefore, for some $\sigma \geq \tau \in \Gamma$, the set P_σ meets more than one member of $\mathfrak{C}$. For such a σ, and for each $x < m(\sigma) \in \Delta$, let

$$P_{\sigma,x} := \{a \in \Delta^\Gamma; \mathrm{dif}(a, m) = \sigma \text{ and } a(\sigma) = x\}.$$

If $\sigma > \tau$ then $g \leq a$ for all $a \in P_{\sigma,x}$, and $P_{\sigma,x} \subseteq P_\sigma$; this is not the case when $\sigma = \tau$. Suppose (again falsely, as we shall see) that for each $x < m(\sigma) \in \Delta$, there is $B_x \in \mathfrak{C}$ such that $P_{\sigma,x} \subseteq B_x$. We note that $P_{\sigma,x} \neq \emptyset$, and hence for each such x, B_x is unique. Then $(-\infty, m(\sigma)) \subseteq \Delta$ and we have a mapping $\psi : (-\infty, m(\sigma)) \to \mathfrak{C}$ defined by $\psi(x) = B_x$. We will again derive a contradiction to Lemma 5.3 by showing that ψ preserves order and has a non-trivial interval in its image. If $x_1 < x_2 < m(\sigma)$, and $r_i \in P_{\sigma,x_i}$, then $r_1 < r_2$

and since $r_i \in B_{x_i}$, we must have $\psi(x_1) = B_{x_1} \leq B_{x_2} = \psi(x_2)$. Hence ψ preserves order. To see that the image of ψ contains a non-trivial interval, we know that P_σ meets more than one member of $\mathfrak{C}$. Let us suppose that $K_1, K_2 \in \mathfrak{C}$, $K_1 \cap P_\sigma \neq \emptyset \neq K_2 \cap P_\sigma$, and $K_1 < K_2$. We show that the interval $(K_1, K_2) \subseteq \mathfrak{C}$ is contained in the image of ψ. Let $T \in \mathfrak{C}$ and $K_1 < T < K_2$. Choose $t \in T$ and $k_i \in K_i \cap P_\sigma$. Then $g \leq k_1 < t < k_2 < m$, and $k_1(\sigma) \leq t(\sigma) \leq k_2(\sigma) < m(\sigma)$. It follows that $\mathrm{dif}(t, m) = \sigma$, and so $\psi(t(\sigma)) = T$. As before, we have a contradiction to Lemma 5.3. Therefore, our assumption about the existence of B_x is wrong, and so there must exist $x < m(\sigma)$ such that $P_{\sigma,x}$ meets two distinct members $A, B \in \mathfrak{C}$. Choose any $a' \in P_{\sigma,x}$ and let $a = a'|_{(-\infty,\sigma]}$. Then $P_{\sigma,x} = N_{a,\sigma}$. Hence $N_{a,\sigma} \cap A \neq \emptyset \neq N_{a,\sigma} \cap B$. Moreover, since $N_{a,\sigma} < m < C$, then $N_{a,\sigma} \cap C = \emptyset$. Thus, Lemma 5.4 is proved.

6 The chains $\Delta^{\mathbb{R}}$

This section is devoted to the proof of the following theorem:

Theorem 6.1 *Let Δ be a countable ordinal ≥ 2, with its least element 0 as base point. Then $\Delta^{\mathbb{R}}$ (with its minimum element deleted) is 2-homogeneous.*

Throughout the section, Δ will denote a countable ordinal ≥ 2 (with base point 0). For $\gamma \in \mathbb{R}$, we remind the reader of the equivalence relations $\mathcal{C}^\gamma$ and $\mathcal{C}_\gamma$ on $\Delta^{\mathbb{R}}$ defined just prior to 2.16.

For $s \in \Delta^{\mathbb{R}}$, $\gamma \in \mathbb{R}$ and $\delta \in \Delta$ define an element $s_{\gamma\delta}$ by setting

$$s_{\gamma\delta}(\gamma') = \begin{cases} s(\gamma') & \text{if } \gamma' < \gamma \\ \delta & \text{if } \gamma' = \gamma \\ 0 & \text{otherwise.} \end{cases}$$

The following is clear.

Proposition 6.2 *For each $s \in \Delta^{\mathbb{R}}$ and $\gamma \in \mathbb{R}$, $s\mathcal{C}^\gamma = \sum_{\delta \in \Delta} s_{\gamma\delta}\mathcal{C}_\gamma \simeq \Delta^{1+(\gamma,\infty)}$, $s\mathcal{C}_\gamma \simeq \Delta^{\mathbb{R}}$, and $s\mathcal{C}^\gamma \simeq \Delta \, \vec{\amalg} \, \Delta^{\mathbb{R}}$.*

Below, Φ denotes $\Delta^{\mathbb{R}}$, which has a minimum (namely $\mathbf{0}$) but no maximum. For $s \in \Phi$, s^+ denotes $\{t \in \Phi \,;\, t > s\}$, and dually for s^-. ${}^\circ\Phi$ denotes Φ with $\mathbf{0}$ deleted. (But all references to $\mathcal{C}^\gamma$-classes and $\mathcal{C}_\gamma$-classes will be to congruences of Φ rather than of ${}^\circ\Phi$, and similarly for s^+ and s^-.) Each class $s\mathcal{C}_\gamma \simeq \Phi$.

The plan of the proof of Theorem 6.1 is to show that for $s \in \Phi$ with $s \neq \mathbf{0}$, s^- and s^+ are both independent (up to isomorphism) of the choice of s, which will give 1-homogeneity for ${}^\circ\Phi$. Indeed, it will turn out that each $s^- \simeq \Phi$ and each $s^+ \simeq {}^\circ\Phi$. Then 2-homogeneity will follow easily.

Since the base point 0 is the minimum element of Δ we have:

Lemma 6.3 *For $t, s \in \Phi$:*

(1) If $t < s$, then $\mathrm{dif}(t, s) \in \mathrm{support}\,(s)$.

(2) If $\gamma_1 < \gamma_2 \in \mathrm{support}\,(s)$, then $s_{\gamma_1 0}\mathcal{C}_{\gamma_1} < s_{\gamma_2 0}\mathcal{C}_{\gamma_2}$.

Proposition 6.4 $\Phi + \Phi \simeq \Phi$.

Proof: Let s be the characteristic function of $\mathbb{N} \subset \mathbb{R}$ defined by

$$s(\gamma) = \begin{cases} 1 & \text{for } \gamma \in \mathbb{N} \subset \mathbb{R}, \\ 0 & \text{otherwise.} \end{cases}$$

We first claim that $s^- \simeq \omega \vec{\amalg} \Phi$. Then support $(s) = \mathbb{N}$. Each $\gamma \in \mathbb{N}$ makes a contribution Ψ_γ to s^- consisting of those $t \in \Phi$ such that $\mathrm{dif}(t, s) = \gamma$ and $t(\gamma) < s(\gamma)$ (so that $t(\gamma) = 0$); and $\gamma_1 < \gamma_2 \in \mathbb{N}$ implies $\Psi_{\gamma_1} < \Psi_{\gamma_2}$. (See Lemma 6.3.) That is,

$$s^- = \sum_{\gamma \in \mathbb{N}} \Psi_\gamma = \sum_{\gamma \in \mathbb{N}} s_{\gamma 0} \mathcal{C}_\gamma.$$

Hence

$$\Phi = s^- + \{s\} + s^+ \simeq (\omega \vec{\amalg} \Phi) + \{s\} + s^+.$$

Now

$$\Phi + \Phi \simeq \Phi + (\omega \vec{\amalg} \Phi) + \{s\} + s^+ \simeq (1 + \omega) \vec{\amalg} \Phi + \{s\} + s^+ \simeq (\omega \vec{\amalg} \Phi) + \{s\} + s^+ \simeq \Phi.$$

□

Lemma 6.5 *The orbits of Aut(Φ) are coterminal in* $^\circ\Phi$ *(except for* $\{\mathbf{0}\}$*).*

Proof: For $\gamma \in \mathbb{R}$ let 1_γ denote the characteristic function of $\{\gamma\}$:

$$1_\gamma(\gamma') = \begin{cases} 1 & \text{if } \gamma' = \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\gamma_1,\ \gamma_2 \in \mathbb{R}$, and $\sigma \in \mathrm{Aut}(\mathbb{R})$ such that $\sigma(\gamma_1) = \gamma_2$. By Lemma 4.1, σ lifts to an automorphism $\hat{\sigma} \in \mathrm{Aut}(\Phi)$ satisfying $\hat{\sigma}(1_{\gamma_1}) = 1_{\gamma_2}$. Hence the set $\{1_\gamma\ ;\ \gamma \in \mathbb{R}\}$, which is coterminal in $^\circ\Phi$, is contained in a single orbit of $\mathrm{Aut}(\Phi)$. Therefore *every* orbit of $\mathrm{Aut}(\Phi)$ is coterminal in $^\circ\Phi$ (except for $\{\mathbf{0}\}$). □

Lemma 6.6 $\omega \vec{\amalg} \Phi \simeq \Phi$ *and* $\omega^* \vec{\amalg} \Phi \simeq {}^\circ\Phi$.

Proof: We prove the first statement, the proof of the second being similar. Fix two isomorphic copies Φ_1 and Φ_2 of Φ. By Proposition 6.4, $\Phi_1 + \Phi_2 \simeq \Phi$. We see from this and Lemma 6.5 that for any $t \in \Phi$, there exists an isomorphism $\psi : \Phi_1 + \Phi_2 \simeq \Phi$ such that $t \in \psi(\Phi_1)$. (Follow any isomorphism from $\Phi_1 + \Phi_2$ to Φ by an appropriate automorphism of Φ.)

Since Φ has cofinality $\aleph_0$, this observation enables us to lay out in Φ a convex partition whose cells are copies of Φ and for which the ordered set of cells is isomorphic to ω. This is done as follows. Let $\{t_i\}_{i \in \omega}$ be a strictly increasing cofinal subset of Φ. By the discussion above, we have an isomorphism $\psi_0 : \Phi_1 + \Phi_2 \simeq \Phi$ such that $t_0 < \psi_0(\Phi_2)$. Proceeding by induction, suppose for each $i = 0, 1, 2, \ldots, n$ we have isomorphisms $\psi_i : \Phi_1 \simeq \psi_i(\Phi_1) \subseteq \Phi$ such that $K_n := \psi_0(\Phi_1) + \psi_2(\Phi_1) + \cdots + \psi_n(\Phi_1)$ is an initial segment of Φ, $\Phi = K_n + K_n^*$, $K_n^* \simeq \Phi$, and $t_n < K_n^*$. Then there is an isomorphism $\psi_{n+1} : \Phi_1 \simeq \psi_{n+1}(\Phi_1) \subseteq K_n^*$ such that $t_{n+1} < \psi_{n+1}(\Phi_2)$. Then $\cap_n K_n^* = \emptyset$, and so $\Phi = \psi_0(\Phi_1) + \psi_1(\Phi_1) + \cdots \simeq \omega \vec{\amalg} \Phi$. □

Lemma 6.7 *For all countable ordinals $\tau \neq 0$, $\tau \vec{\amalg} \Phi \simeq \Phi$.*

Proof: If τ is finite, $\tau \vec{\amalg} \Phi \simeq \Phi$ by Proposition 6.4. Suppose now that τ is infinite. Write $\tau = \sum_{i \in I} \tau_i + p$, where $I \simeq \omega$, $0 \neq \tau_i < \tau$ for each $i \in I$, and p is finite. By induction, and with the aid of Lemma 6.6,

$$\begin{aligned} \tau \vec{\amalg} \Phi &\simeq ((\sum_{i \in I} \tau_i) + p) \vec{\amalg} \Phi \simeq \sum_{i \in I} (\tau_i \vec{\amalg} \Phi) + p \vec{\amalg} \Phi \simeq \sum_{i \in I} \Phi + p \vec{\amalg} \Phi \\ &\simeq \omega \vec{\amalg} \Phi + p \vec{\amalg} \Phi \simeq \Phi + p \vec{\amalg} \Phi \simeq \Phi. \end{aligned}$$

$\square$

As promised, we have

Proposition 6.8 *For each $s \in \Phi$ (except for $s = \mathbf{0}$), $s^- \simeq \Phi$.*

Proof: The proof parallels closely that of Proposition 6.4. Let $s \in \Phi$ with $s \neq \mathbf{0}$. Then support (s) is a nonempty countable wellordered subset of $\mathbb{R}$, and we denote it by I. As in 6.4, each $\gamma \in I$ makes a contribution Ψ_γ to s^- consisting of those $t \in \Phi$ such that $\mathrm{dif}(t, s) = \gamma$ and $t(\gamma) < s(\gamma)$. We have

$$s^- = \sum_{\gamma \in I} \Psi_\gamma = \sum_{\gamma \in I} (\sum_{\delta < s(\gamma)} s_{\gamma\delta} \mathcal{C}_\gamma) \simeq \sum_{\gamma \in I} (\sum_{\delta < s(\gamma)} \Phi) \simeq \sum_{\gamma \in I} \Phi \simeq \Phi \,,$$

where the last two steps follow from Lemma 6.7 since Δ and I are countable. $\square$

Now we turn to s^+, with $^\circ\Phi \vec{\amalg} \Phi$ (which will turn out to be isomorphic to $^\circ\Phi$) playing the role played for s^- by Φ. The next two results are consequences of Lemma 6.7.

Corollary 6.9 *Each $\mathcal{C}^\gamma$-class (as well as each $\mathcal{C}_\gamma$-class) is isomorphic to Φ.*

Lemma 6.10 $\Delta^{\mathbb{R}^{<0}} \vec{\amalg} \Delta \vec{\amalg} \Delta^{\mathbb{R}^{>0}} \simeq \Phi \vec{\amalg} \Delta \vec{\amalg} \Phi \simeq \Phi \vec{\amalg} \Phi.$

Proposition 6.11 $\Phi \vec{\amalg} \Phi \simeq \Phi.$

Proof: By Lemma 6.10 $\Phi \vec{\amalg} \Phi \simeq \Delta^{\mathbb{R}^{<0}} \vec{\amalg} \Delta \vec{\amalg} \Delta^{\mathbb{R}^{>0}} \simeq \Delta^{\mathbb{R}} = \Phi.$ $\square$

Corollary 6.12 $\Phi + (^\circ\Phi \vec{\amalg} \Phi) \simeq \Phi.$

Proof: Since $\Phi \vec{\amalg} \Phi \simeq \Phi$, then $(\{0\} + {^\circ\Phi}) \vec{\amalg} \Phi \simeq \Phi$, so $(\{0\} \vec{\amalg} \Phi) + (^\circ\Phi \vec{\amalg} \Phi) \simeq \Phi.$ $\square$

Corollary 6.13 $^\circ\Phi + (^\circ\Phi \vec{\amalg} \Phi) \simeq {^\circ\Phi}.$

Corollary 6.14 $\omega^* \vec{\amalg} (^\circ\Phi \vec{\amalg} \Phi) \simeq {^\circ\Phi}.$

Proof: Since $^\circ\Phi$ has countable coinitiality and the orbits of $\mathrm{Aut}(^\circ\Phi)$ are coinitial in $^\circ\Phi$, the proof of this corollary is almost identical to that of Lemma 6.6, using Corollary 6.13 instead of Lemma 6.4. $\square$

Corollary 6.15 $^\circ\Phi \vec{\amalg}\, \Phi + {}^\circ\Phi \vec{\amalg}\, \Phi \simeq {}^\circ\Phi \vec{\amalg}\, \Phi$.

Proof: By Proposition 6.4 $\Phi + \Phi \simeq \Phi$. Lexing by $^\circ\Phi$ establishes the result. $\square$

Corollary 6.16 $\omega^* \vec{\amalg}\, ({}^\circ\Phi \vec{\amalg}\, \Phi) \simeq {}^\circ\Phi \vec{\amalg}\, \Phi$.

Proof: The proof parallels that of Corollary 6.14, using Corollary 6.15 instead of Corollary 6.13. Coinitiality of orbits holds for $\mathrm{Aut}({}^\circ\Phi \vec{\amalg}\, \Phi)$ since it holds for $\mathrm{Aut}({}^\circ\Phi)$. $\square$

Lemma 6.17 $^\circ\Phi \vec{\amalg}\, \Phi \simeq {}^\circ\Phi$.

Proof: This follows from Corollary 6.16 and Corollary 6.14. $\square$

Corollary 6.12 and Lemma 6.17 yield

Proposition 6.18 $\Phi + {}^\circ\Phi \simeq \Phi$.

Corollary 6.19 $^\circ\Phi + {}^\circ\Phi \simeq {}^\circ\Phi$.

In parallel with Lemma 6.7 we have

Lemma 6.20 *For all countable ordinals $\tau \neq 0$,*

$$\tau^* \vec{\amalg}\, \Phi \simeq \begin{cases} \Phi & \textit{if } \tau \textit{ is a successor ordinal,} \\ {}^\circ\Phi & \textit{if } \tau \textit{ is a limit ordinal.} \end{cases}$$

This remains true even if the local copies of Φ are replaced by copies of $^\circ\Phi$ throughout an arbitrary subset of the index chain τ^* (except that if τ^* has a smallest element and the smallest copy of Φ is replaced by $^\circ\Phi$, the result is isomorphic to $^\circ\Phi$).

Proof: We prove the special case first. If τ is finite, $\tau^* \vec{\amalg}\, \Phi \simeq \Phi$ by Proposition 6.4. Suppose now that τ is infinite. Write $\tau = \sum_{i\in I} \tau_i + p$, where $I \simeq \omega$, $0 \neq \tau_i < \tau$ for each $i \in I$ and each τ_i is a successor ordinal, and p is finite. By induction on τ, and with the aid of Lemma 6.6 and Proposition 6.18,

$$\begin{aligned} \tau^* \vec{\amalg}\, \Phi &\simeq ((\sum_{i\in I} \tau_i) + p)^* \vec{\amalg}\, \Phi \simeq p^* \vec{\amalg}\, \Phi + \sum_{i\in I^*} \tau_i^* \vec{\amalg}\, \Phi \simeq p^* \vec{\amalg}\, \Phi + \sum_{i\in I^*} \Phi \\ &\simeq p^* \vec{\amalg}\, \Phi + \omega^* \vec{\amalg}\, \Phi \simeq p^* \vec{\amalg}\, \Phi + {}^\circ\Phi \simeq \begin{cases} \Phi & \text{if } p \neq 0, \\ {}^\circ\Phi & \text{if } p = 0. \end{cases} \end{aligned}$$

To deduce the more general assertion from the special case, we explain how to obtain an isomorphism from $\tau^* \vec{\amalg}\, \Phi$ onto its modification. Decompose each local copy of Φ as the sum $\Phi = \Phi' + \Phi''$ of two copies of Φ. Then given any sum $\Phi_1 + \Phi_2$ of consecutive local copies of Φ in $\tau^* \vec{\amalg}\, \Phi$, we have $\Phi_1 + \Phi_2 \simeq \Phi_1' + (\Phi_1'' + \Phi_2') + \Phi_2'' \simeq \Phi_1' + (\Phi_1'' + {}^\circ\Phi_2') + \Phi_2''$ (by Propositions 6.18 and 6.15) $\simeq \Phi_1 + {}^\circ\Phi_2$; and indeed via an isomorphism which is the identity on $\Phi_1' \cup \Phi_2''$, which permits the isomorphisms for the various sums $\Phi_1 + \Phi_2$ to be spliced together. $\square$

Again as promised, we have

Proposition 6.21 *For each $s \in \Phi$, $s^+ \simeq {}^\circ\Phi$.*

Proof: The proof parallels that of Proposition 6.8, but with some twists. Let $s \in \Phi$. We may assume $s \neq \mathbf{0}$ since if $s = \mathbf{0}$ then $s^+ = {}^\circ\Phi$ by definition. Let $I = \text{support}\,(s)$. As in Proposition 6.8 , each $\gamma \in I$ makes a contribution $\Psi_\gamma = \Psi'_\gamma + \Psi''_\gamma$ (to be described below) to s^+, and

$$s^+ \simeq \Psi_\infty + \sum_{\gamma \in I^*} \Psi_\gamma \ ,$$

where the extra term Ψ_∞ will also be described below.

We now describe Ψ_γ. Its first term Ψ'_γ consists of those $t \in \Phi$ such that $\text{dif}(t, s) = \gamma$ and $t(\gamma) > s(\gamma)$. Its second term Ψ''_γ consists of those $t \in \Phi$ such that $\text{dif}(t, s)$ is slightly less than γ, in the following sense:

(1) If γ has an immediate predecessor β in I, $\text{dif}(t, s) \in (\beta, \gamma) \subset \mathbb{R}$.

(2) If $\gamma = \min I$, $\text{dif}(t, s) \in (-\infty, \gamma)$.

(3) If $\{\eta \in I\ ;\ \eta < \gamma\}$ has no largest element, and if β is its supremum in $\mathbb{R}$, then
  (a) If $\beta < \gamma$, $\text{dif}(t, s)$ is to be in $[\beta, \gamma)$.
  (b) If $\beta = \gamma$, $\Psi''_\gamma = \emptyset$.

Observe that $\Psi'_\gamma < \Psi''_\gamma$ since $\Delta^{\mathbb{R}}$ is ordered lexicographically from the left. Similarly $\gamma_1 < \gamma_2 \in I$ implies $\Psi_{\gamma_2} < \Psi_{\gamma_1}$. This is why in writing $s^+ \simeq \Psi_\infty + \sum_{\gamma \in I^*} \Psi_\gamma$, the summation is over I^*, not I.

Finally, Ψ_∞ consists of those $t \in \Phi$ such that $\text{dif}(t, s) > I$:

(1) If I has a largest element β, $\text{dif}(t, s) \in (\beta, \infty)$.

(2) If I has no largest element, then
  (a) If I is not cofinal in $\mathbb{R}$ and $\beta = \sup I$, $\text{dif}(t, s) \in [\beta, \infty)$.
  (b) If I is cofinal in $\mathbb{R}$, $\Psi_\infty = \emptyset$.

Then $\Psi_\infty < \Psi_\gamma$ for all γ.

Since s^+ is partitioned by Ψ_∞ and the Ψ_γ ' s, we have

$$s^+ = \Psi_\infty + \sum_{\gamma \in I^*} \Psi_\gamma = \Psi_\infty + \sum_{\gamma \in I^*} (\Psi'_\gamma + \Psi''_\gamma) \ .$$

Now we write $\Delta^\gamma := \{\delta \in \Delta\ ;\ \delta > s(\gamma)\}$. Then by Corollary 6.9

$$\Psi'_\gamma \simeq \Delta^\gamma \,\vec{\Pi}\, \Delta^{(\gamma,\infty)} \simeq \Delta^\gamma \,\vec{\Pi}\, \Phi \simeq \Phi$$

(unless $s(\gamma)$ is the maximum element of Δ, in which case $\Psi'_\gamma = \emptyset$).

Using Corollary 6.9 and Lemma 6.17, and with reference to the several cases in the above description of Ψ''_γ, we get

$$\Psi''_\gamma \simeq \begin{cases} {}^\circ(\Delta^{(\beta,\gamma)}) \,\vec{\Pi}\, \Delta^{\mathbb{R}^{\geq\gamma}} \simeq {}^\circ\Phi \,\vec{\Pi}\, \Phi \simeq {}^\circ\Phi & \text{in case (1) ,} \\ {}^\circ(\Delta^{\mathbb{R}^{<\gamma}}) \,\vec{\Pi}\, \Delta^{\mathbb{R}^{\geq\gamma}} \simeq {}^\circ\Phi \,\vec{\Pi}\, \Phi \simeq {}^\circ\Phi & \text{in case (2) ,} \\ {}^\circ(\Delta^{[\beta,\gamma)}) \,\vec{\Pi}\, \Delta^{\mathbb{R}^{\geq\gamma}} \simeq {}^\circ(\Delta^{\mathbb{R}^{\geq\beta}}) \,\vec{\Pi}\, \Phi \simeq {}^\circ\Phi \,\vec{\Pi}\, \Phi \simeq {}^\circ\Phi & \text{in case (3a) ,} \\ \emptyset & \text{in case (3b) .} \end{cases}$$

Thus in each case, $\Psi''_\gamma \simeq {}^\circ\Phi$ or $\Psi''_\gamma \simeq \emptyset$.
Since $\Phi + {}^\circ\Phi \simeq \Phi$ by Proposition 6.18, we have for each γ that Ψ_γ is isomorphic to Φ or ${}^\circ\Phi$ or $\emptyset$. Moreover, since $\Psi_\gamma = \emptyset$ is possible only in case (3b) of Ψ''_γ, which forces γ to correspond to a limit ordinal in I, we may assume that Ψ_γ is isomorphic to Φ or ${}^\circ\Phi$ (replacing I by $\{\gamma \in I\,;\, \Psi_\gamma \neq \emptyset\}$).
An analysis similar to that for Ψ''_γ shows that either $\Psi_\infty = \emptyset$ (if I is cofinal in $\mathbb{R}$), or $\Psi_\infty \simeq {}^\circ\Phi$.

Putting all this together, we have

$$s^+ \simeq \Psi_\infty + \sum_{\gamma \in I^*} \Psi_\gamma \simeq {}^\circ\Phi + \sum_{\gamma \in I^*} \Psi_\gamma,$$

where the first term ${}^\circ\Phi$ is missing if and only if I is cofinal in $\mathbb{R}$, and where for each γ, $\Psi_\gamma \simeq \Phi$ or $\Psi_\gamma \simeq {}^\circ\Phi$. Then Lemma 6.20 guarantees that $s^+ \simeq {}^\circ\Phi$ in all cases; for if the first term ${}^\circ\Phi$ is missing, then I is cofinal in $\mathbb{R}$ and thus has no largest element, so that I^* has no smallest element, making $\sum_{\gamma \in I^*} \Psi_\gamma \simeq {}^\circ\Phi$. $\square$

We can now prove Theorem 6.1.
Proof: The homogeneity of ${}^\circ\Phi$ follows from Propostions 6.8 and 6.21. 2-homogeneity then follows from the fact that all open intervals (s,t) of ${}^\circ\Phi$ are isomorphic, which holds because always $s^+ \simeq {}^\circ\Phi$ and then $(s,t) \simeq {}^\circ\Phi$ by an application of Proposition 6.8 to $t \in s^+$. $\square$

Example 6.22 Theorem 6.1 is not true without the countability hypothesis on Δ: let $\Delta = \omega_1 + 1$. The last element of Δ has uncountable left character. For $\delta \in \Delta$, define $s_\delta \in \Delta^{\mathbb{R}}$ by setting

$$s_\delta(\gamma) = \begin{cases} \delta & \text{if } \gamma = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For δ limit ordinal, the left character of s_δ in $\Delta^{\mathbb{R}}$ coincides with the left charater of δ in Δ. Hence $\Delta^{\mathbb{R}}$ has points of uncountable (and also countable) left characters, and hence ${}^\circ\Delta^{\mathbb{R}}$ cannot be homogeneous, much less 2-homogeneous.

The following problems arise:

Problem 1 : We do not know how many different chains we are talking about in Theorem 6.1. That is, we do not know when two such lexicographic powers are isomorphic (for different countable ordinals Δ) – possibly always.

Problem 2 : If $\Delta = n$ is finite, then by Corollary 2.5 $\Delta^{\mathbb{R}} \simeq (\Delta^{\mathbb{R}})^*$, with left hand side computed in base 0, and right hand side computed in base $n-1$. So $\Delta^{\mathbb{R}}$ (with its maximum element deleted) computed in base $n-1$ is 2-homogeneous as well.
But in general, we do not know whether Theorem 6.1 is true if the chosen base point is not the least element 0 of Δ.

Closely related to Problem 2 are the following.

Problem 3 : For Δ as in Theorem 6.1, we do not know whether the lexicographic powers computed in different base points yield isomorphic chains (after their endpoints are deleted).

Problem 4 : We do not know whether the 2-homogeneous chains $\Delta^{\mathbb{R}}$ appearing in Theorem 6.1 (computed in base 0) are symmetric (after their minimum elements are deleted). Note that a positive solution for the special case $\Delta = n$ finite requires proving that $\Delta^{\mathbb{R}}$ computed in base 0 is isomorphic to $\Delta^{\mathbb{R}}$ computed in base $n-1$ (after the endpoints are deleted).

In [H1] and [H2] Hausdorff was interested in coterminalities and in characters of points and holes in lexicographic powers. (The characters of a hole $\overline{s}$ in a chain A are those of $\overline{s}$ considered as a point in $\overline{A}$). In the present paper, point and hole characters have been barely mentioned. This is because in most cases that we considered, all characters and coterminalities are at most countable. (This is due to the well known fact, not difficult to show, that if Δ and Γ both have all characters and coterminalities at most countable, then so does Δ^{Γ}; see Proposition 2.19). This produces the following slightly stronger conclusion for Theorem 6.1, also of interest to Hausdorff. Below, $X := {}^{\circ}(\Delta^{\mathbb{R}})$.

Proposition 6.23 *Assume the hypothesis of Theorem 6.1. Then all open intervals of X, with endpoints now permitted to come from $\overline{X} \cup \{+\infty, -\infty\}$, are isomorphic. Also the holes of X form a single orbit in the action of Aut(X) on $\overline{X}$, and $\overline{X}$ is 2-homogeneous.*

Proof: Patching (cf. Proposition 2.20), which works because X is 2-homogeneous and all its characters and coterminalities are countable. To say that the holes of X form a single orbit, we need to show that there are holes; this is addressed below. $\square$

Under the hypothesis of Theorem 6.1, $\Delta^{\mathbb{R}}$ is never Dedekind complete - the $\mathcal{C}^{\gamma}$-classes do not have suprema in $\Delta^{\mathbb{R}}$. (In fact, the suprema are the *only* holes in $\Delta^{\mathbb{R}}$, but we omit the proof).

Problem 5 : Is $\overline{X} \setminus X \simeq X$?

Example 6.24 The chain $(\omega_1 + 1)^{\mathbb{R}}$ considered in Example 6.22 cannot be isomorphic to any $\Delta^{\mathbb{R}}$ satisfying the hypothesis of Theorem 6.1, due to the discrepancies in point characters (alternately, to discrepancies in 2-homogeneity). Moreover $\omega_1^{\mathbb{R}}$ is isomorphic to neither since its points all have at most countable characters, but it does have holes of uncountable left character (the suprema of $\mathcal{C}^{\gamma}$-classes).

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