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A Note on Extrema of Linear Combinations of Elementary Symmetric Functions

Alexander Kovačec[a] , Salma Kuhlmann[b] and Cordian Riener[c]

[a]*Departamento de Matemática, Faculdade de Ciências e Tecnologia, Apartado 3008, EC Universidade, P 3001-454 Coimbra, Portugal*; [b]*Fachbereich Mathematik und Statistik, Universität Konstanz, 78457 Konstanz, Germany*; [c]*Institut für Mathematik, Goethe Universität, 60325 Frankfurt, Germany*

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This note provides a new approach to a result of Foregger [2] and related earlier results by Keilson [4] and Eberlein [1]. Using quite different techniques, we prove a more general result from which the others follow easily. Finally, we argue that the proof in [2] is flawed.

Keywords: Extrema, elementary symmetric functions, hyperbolic polynomials

Throughout this note write $E_j(x) = E_j(x_1, ..., x_n)$, $j = 0, 1, \ldots, n$, for the j-th elementary symmetric polynomial in n variables. Let $H = H_\gamma = \{x \in \mathbb{R}^n : E_1(x) = \gamma\}$ be a hyperplane normal to the vector $1_n = (1, 1, ..., 1) \in \mathbb{R}^n$ and ϕ be a real linear combination of the E_j. Putting $D := H \cap [0, 1]^n$, Keilson [4] investigated the question where the function $\phi : D \to \mathbb{R}$ assumes its extreme values. Eberlein [1] noted that the result of Keilson was actually already known by Chebyshev in 1846 and Hoeffding in 1956. He assumed $0 \le a_i \le b_i \le 1$ and investigated the same question on more general domains $D' = H \cap ([a_1, b_1] \times \cdots \times [a_n, b_n])$. Foregger, interested in solving a problem by Pierce [7], also solved by Li [6], returned to the original Keilson-Chebyshev question, and proposed to show that if ϕ is non-constant on D (i.e. has degree at least two) and attains its extremum at an interior point of D, then this point must be the symmetric point of D, that is $\frac{\gamma}{n} 1_n$. At the end of his paper Foregger invites to establish similar results to his for Eberlein's domains.

In this note, we establish below in Theorem 1, Theorem 3 and Corollary 5 results from which the above mentioned are easily derived. Our method, originally developed by Riener [8] to provide an algebraic approach to Timofte's results [10] is different and much simpler: it is based on the observation that the elementary symmetric functions are closely related to roots of univariate *hyperbolic polynomials*, i.e. polynomials with only real roots. Given a function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, write loc.extr$(f)$ for the set of local extrema of f; that is for the set of all points u in D admitting a neighborhood $N = N_u \subseteq D$ such that $\forall x \in N \; f(x) \ge f(u)$ or $\forall x \in N \; f(x) \le f(u)$.

Theorem 1 : *If a real linear combination ϕ of elementary symmetric polynomials has degree at least two, then it has as a function $\phi : H \to \mathbb{R}$ on the hyperplane $H = \{x : E_1(x) = \gamma\}$ at most one local extremum; more precisely* loc.extr$(\phi) \subseteq \{\frac{\gamma}{n} 1_n\}$.

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We will need an auxiliary lemma.

Lemma 2: *Let $f \in \mathbb{R}[t]$ be hyperbolic of degree n with only $s \in \{1, ..., n-1\}$* distinct *(real) roots all lying in some open interval I. Then there exists $g \in \mathbb{R}[t]$ of degree $n-s$ so that for all small $\varepsilon > 0$, the polynomials $f + \varepsilon g$ and $f - \varepsilon g$ are hyperbolic and have both more than s distinct roots all lying in I.*

Proof: Let $x_1, \ldots, x_s$, be the distinct zeros of f which by hypothesis all lie in I. Without loss of generality assume I bounded and for notational convenience assume $I =]x_0, x_{s+1}[$ and $x_0 < x_1 < \ldots < x_s < x_{s+1}$. We have a factorization

$$f = \prod_{i=1}^{s}(t - x_i) \cdot g_1 =: p \cdot g_1,$$

where the roots of g_1 are all in $\{x_1, \ldots, x_s\}$ and g_1 is of degree $n - s \geq 1$. Now for $x_i < t < x_{i+1}$, $i = 0, \ldots, s$, $\operatorname{sgn} p(t) = (-1)^{s-i}$. Therefore there exist points $\xi_i \in]x_i, x_{i+1}[$, and $\varepsilon > 0$, such that if $s - i$ is odd, $p(\xi_i) < -2\varepsilon$, while if $s - i$ is even, $p(\xi_i) > 2\varepsilon$. Therefore the polynomials $p \pm \varepsilon$ have in $\xi_0, \xi_1, \xi_2, ..., \xi_{s-1}, \xi_s$ alternating signs. So they have in each of the intervals $]\xi_i, \xi_{i+1}[$, $i = 0, 1, ..., s-1$, a root. Therefore $p \pm \varepsilon$ have both s distinct real roots and so are hyperbolic and the roots all lie in I. Furthermore we see that $p \pm \varepsilon$ have none of their roots in the set $\{x_1, ..., x_s\}$. Hence $(p \pm \varepsilon) \cdot g_1 = f \pm \varepsilon g_1$ are hyperbolic and have each more than s distinct roots all lying in I. □

Proof: [Proof of theorem 1] If the set of local extrema of $\phi : H \to \mathbb{R}$ is nonempty, choose a local extremum, say $a := (a_1, \cdots, a_n)$, so that the number $s := |\{a_1, \ldots, a_n\}|$ of distinct components of a is maximal and consider the univariate polynomial

$$f_a := \prod_{i=1}^{n}(-a_i + t) = \sum_{i=0}^{n-2}(-1)^{n-i}E_{n-i}(a)t^i - \gamma t^{n-1} + t^n.$$

Here we used Viète's formula and to facilitate reading of the following we think of f_a and similar univariate polynomials as written from left to right with rising powers and write ϕ as $\phi(x) = \sum_{i=0}^{n-2} c_{n-i}E_{n-i}(x) + c_1E_1(x) + c_0E_0(x)$. Assume $s \in \{2, 3, ..., n-1\}$. By lemma 2 there exists a polynomial $g = \sum_{i=0}^{n-s} g_{n-i}t^i$ of degree $n-s$, and an $\varepsilon_0 > 0$ such that for all ε with $0 < |\varepsilon| < \varepsilon_0$, the polynomial $f_a + \varepsilon g$ is hyperbolic and has at least $s+1$ distinct roots. We can represent for every ε the n (real) roots of $f_a + \varepsilon g$ as the entries of $a(\varepsilon) = (a_1(\varepsilon), ..., a_n(\varepsilon))$, and according to Kato [3, p.109] think of $a(.)$ actually as a continuous function. We then have a similar equation as above for $f_a + \varepsilon g$ (in place of f_a) and $E_{n-i}(a(\varepsilon))$ (in place of $E_{n-i}(a)$). We find $(-1)^{n-i}E_{n-i}(a(\varepsilon)) = (-1)^{n-i}E_{n-i}(a) + \varepsilon g_{n-i}$, where for $i > n-s$, we put $g_{n-i} = 0$. In particular $E_1(a(\varepsilon)) = \gamma$. By these formulae we find

$$\phi(a(\varepsilon)) = \phi(a) + \varepsilon \sum_{i=0}^{n-s}(-1)^{n-i}c_{n-i}g_{n-i}.$$

It now follows for all ε of small modulus and appropriate sign, that $\phi(a(-\varepsilon)) \leq \phi(a) \leq \phi(a(\varepsilon))$ with $a(\pm\varepsilon) \in O_a \subseteq H$, O_a a neighborhood of a. So in every neighborhood of a there are points $a(-\varepsilon)$ at which ϕ is not larger than $\phi(a)$ and $a(\varepsilon)$ at which ϕ is not smaller than $\phi(a)$ and which have at least $s+1$ distinct coordinates. This contradicts our choice of a. Therefore $s \in \{1, n\}$, that is, f_a has either one root of multiplicity n, or n distinct roots. Assume $s = n$. Since degree$(\phi) \geq 2$, $(c_2, ..., c_n) \neq 0$. Thus there exist reals $g_2, ..., g_n$ so that $\sum_{i=0}^{n-2} c_{n-i}g_{n-i} \neq 0$. Consider the polynomial $g = \sum_{i=0}^{n-2} g_{n-i}t^i$ and note that $f_a + \varepsilon g$ will have for all ε of small modulus n roots. With the reasoning above, we infer this time strict inequalities $\phi(a(-\varepsilon)) < \phi(a) < \phi(a(\varepsilon))$, again arriving at a contradiction. Hence $s = 1$ and $a = \frac{\gamma}{n}1_n$, as we wished to show. □

A simple adaptation of this proof which we leave to the reader yields the following generalization of theorem 1 which will not be further used in this paper.

Theorem 3: *If the polynomial ϕ of theorem 1 has degree at least $k+1$ and is considered as a function on the real variety $H = \{x : E_1(x) = \gamma_1, ..., E_k(x) = \gamma_k\}$, then each of its local extrema a has at most k distinct components, i.e. $|\{a_1, \ldots, a_n\}| \leq k$.*

We can now derive a corollary in the spirit of Eberlein and Foregger which is more complete. The following notation is fixed through the rest of the paper. Let $a_i < b_i$, $i = 1, \ldots, n$ be real numbers. Denote by $e_i = (0, ..., 0, 1, 0, ...0)$, the i-th standard vector in $\mathbb{R}^n$. Using the $(2n+2) \times n$ matrix

$$A = [-e_1^\mathsf{T}, e_1^\mathsf{T}, -e_2^\mathsf{T}, e_2^\mathsf{T}, \ldots, -e_n^\mathsf{T}, e_n^\mathsf{T}, -1_n^\mathsf{T}, 1_n^\mathsf{T}]^\mathsf{T}$$

(whose rows are $-e_1, e_1, -e_2, ...$) and the column

$$d = (-a_1, b_1, -a_2, b_2, \ldots, -a_n, b_n, -\gamma, \gamma)^\mathsf{T},$$

the set $D' = H \cap ([a_1, b_1] \times \cdots \times [a_n, b_n])$ can be defined as $D' = \{x : Ax \leq d\}$. Since D' is bounded it is a polytope in the sense of Schrijver [9, p. 89]. For $K \subseteq \{1, ..., n\}$ we write $K^c = \{1, \ldots, n\} \setminus K$ for its complement. We begin by characterizing the faces of D'.

Lemma 4: *A subset F of $\mathbb{R}^n$ is face of D' if and only if it is nonempty and there exists a subset K in $\{1, 2, \ldots, n\}$ and $u_l \in \{a_l, b_l\}$ for all $l \in K^c$, so that*

$$F = \{\sum_{k\in K} x_k e_k + \sum_{l\in K^c} u_l e_l : \forall k \in K\ a_k \leq x_k \leq b_k \text{ and } \sum_{k\in K} x_k + \sum_{k\in K^c} u_k = \gamma\}.$$

If K^c is a maximal set such that the inequality conditions at the right of the ':' are simultaneously satisfiable without equalities, then the affine dimension of F is $(|K| - 1)^+$.

Proof: By [9, p101], a set $F \subseteq \mathbb{R}^n$ is a face of D' iff $F \neq \emptyset$ and $F = \{x : Ax \leq d \text{ and } A'x = d'\}$, where A' and d' are obtained from A, d, respectively, by selecting the same row indices. It may happen that there exists a further, say i-th row $a_{i,*}$ of A, such that $a_{i,*}x = d_i$ for all $x \in F$. We think from now on of all such rows as included in A' and write $A^=$ for this submatrix of A, and $d^=$ for the corresponding subvector of d. Let A^+ be the complementary submatrix of $A^=$ in A. Then for each row index i of A defining a row of A^+ there exists a vector $x^{(i)} \in F$ such that $A^+x^{(i)} < d_i$. The arithmetic mean of all such vectors yields a vector $x \in F$ so that $A^+x < d^+$ (meaning that all component inequalities are strict). We note that F can be written as $F = \{x : A^+x \leq d \text{ and } A^=x = d^=\}$. The set of row indices of A defining $A^=$ contains evidently $\{2n+1, 2n+2\}$ but contains from each of the sets $\{2i-1, 2i\}$, $i = 1, ..., n$ at most one index, since otherwise F would be empty. An index lying in $\{2i-1, 2i\}$ fixes x_i to be equal to some $u_i \in \{a_i, b_i\}$. Let $K^c \subseteq \{1, ..., n\}$ be the set of indices i of variables so obtained, viewed as the complement of some set $K \subseteq \{1, ..., n\}$. Then the components x_k, $k \in K$ of $x \in F$ may satisfy $a_k < x_k < b_k$ and $\sum_{k\in K} x_k = \gamma - \sum_{k\in K^c} u_k$. It is now clear that F can be written as above and the affine dimension of F (which is by definition the dimension of its affine hull) equals $(|K| - 1)^+$. □

Corollary 5: *Assume that as a function, $\phi : D' \to \mathbb{R}$ is nonconstant at every edge of the polytope D'. Then every local extremum p of ϕ is point in a cartesian product $\{a_1, e, b_1\} \times \cdots \times \{a_n, e, b_n\}$. Here e has to be chosen such that the sum of the components of p is γ.*

Proof: Let $p \in D'$ be a local extremum of ϕ. Since the faces of D' form a lattice [9, section 8.6], there exists a unique face F of minimal dimension containing p. Then $p \in \operatorname{int} F$. We think of F as written in the above lemma, with the set K^c there chosen maximal. If $\dim F = 0$, then p is a vertex, $F = \{p\}$, and $|K| = 0$ or $|K| = 1$. It then follows by the above characterization of F that $p = (p_1, \ldots, p_n)$ is a point so that for at least $n-1$ indices i, $p_i \in \{a_i, b_i\}$. Consequently p lies in one of the cartesian products admitted. Assume now $\dim F = k \geq 1$. Then F has a nonempty relative interior given via $\gamma^* = \gamma - \sum_{k \in K^c} u_k$ by

$$\operatorname{int} F = \{\sum_{k \in K} x_k e_k + \sum_{l \in K^c} u_l e_l : \forall k \in K\; a_k < x_k < b_k \text{ and } \sum_{k \in K} x_k = \gamma^*\}.$$

Now F has faces of D' which have dimension 1 [9, section 8.3] on which by assumption ϕ is not constant; consequently ϕ is not constant on the affine hull of F, and hence is a polynomial of degree≥ 2 on it. The local extremum $p \in \operatorname{int} F$ of ϕ is necessarily also local extremum of the restriction of ϕ to the affine hull of F. Now this restriction is $\phi'(x_{k_1}, ..., x_{k_{|K|}}) = \phi'(\sum_{k \in K} x_k e_k) := \phi(\sum_{k \in K} x_k e_k + \sum_{l \in K^c} u_l e_l)$. It is again a certain real linear combination of elementary symmetric functions of variables x_k, $k \in K = \{k_1, ..., k_{|K|}\}$: this follows from that a similar fact holds for each of the elementary symmetric functions. Applying our main result to the affine hull of F, we get that $p = \sum_{k \in K} \frac{\gamma^*}{|K|} e_k + \sum_{l \in K^c} u_l e_l$, as we wished to show. □

Remark 1: a. The mentioned authors state some of their results in somewhat unprecise terms. Thus in [4, p.269] we find the statement that 'ϕ assumes its maximum and minimum on the set D at either its boundary or the symmetric point.' If one analyzes Keilson's proof one finds that he establishes the following: if $a \in D$ is a critical point of ϕ, then a is the symmetric point or there is a line segment in D on which ϕ is constant. It follows that if $a \in \operatorname{int} D$ (the relative interior of D) is a *strict* local extremum (i.e. a point so that for all points $x \in \operatorname{int} D \setminus \{a\}$ lying in a neighbourhood of a, there holds $\phi(x) > \phi(a)$ or for all such x there holds $\phi(x) < \phi(a)$), then a is the symmetric point. For Keilson's reasoning to be convincing his 'extremum' on p. 270, line -12 must be assumed to be strict. Eberlein's theorem 1 [1, p. 312] tells us that the 'minimum and the maximum of ϕ is assumed at least among the points whose components which are not endpoints are all equal. Moreover if the maximum and the minimum is attained only in the interior of D' then it is assumed uniquely at the point $\frac{\gamma}{n} 1_n$'. After a more precise formulation, the corresponding proof which shows similarities with Keilson's is right but unfortunately does not give more. In particular it does not exclude for nonconstant ϕ lines on which ϕ has constant extreme value and which extend to the boundary of D'. Thus again there may be interior extrema other than $\frac{\gamma}{n} 1_n$. This possibility is what Foregger proposed to exclude for his domain D. He attempts establishing his result by induction over dimension via reasoning not easily adaptable to Eberlein's more general domain D'. We now discuss the error in Foregger's proof, and point out another mistake in his paper.

b. (i) Using exactly his notation, in [2, p.384], Foregger derives for $s < n$, $c \in \mathbb{R}^{n-s}$ constant, and $y \in \mathbb{R}^s$, the first two lines in the following chain; the third line is a consequence of noting that $E_0(y) = 1$, and $E_1(y) = \gamma^*$ by Foregger's (local) definition (of) $C_{\gamma^*} = \{y \in \mathbb{R}^s : \sum_{i=1}^s y_i = \gamma^*, y_i \in [0,1]\} \subset \mathbb{R}^s$ on p. 383:

$$\begin{aligned} 0 &= \phi^*(y) \\ &= \sum_{k=0}^{n} E_k(y) \sum_{r=k}^{n} c_r E_{r-k}(c) - \sum_{r=2}^{n} c_r E_r(0,c) \\ &= \sum_{r=0}^{n} c_r E_r(c) + \gamma^* \sum_{r=1}^{n} c_r E_{r-1}(c) - \sum_{r=2}^{n} c_r E_r(0,c) + \sum_{k=2}^{n} E_k(y) \sum_{r=k}^{n} c_r E_{r-k}(c). \end{aligned}$$

Note that $y \in \mathbb{R}^s$, so the definition of the elementary symmetric functions requires $E_{s+1}(y) = \cdots = E_n(y) = 0$, a fact not observed in [2]. Granted that as functions $1, E_2, E_3, ..., E_s$ are (usually) linearly independent on C_{γ^*} - to see this put $n-1$ variables equal to real variable t, the other equal to $\gamma^* - t$ and observe that $E_j(t, ..., t, \gamma^* - t)$ is a polynomial of degree j in t - we may infer the equations $\sum_{r=k}^{n} c_r E_{r-k}(c) = 0, k = 2, 3, ..., s$. The problem in [2] is that these equations are claimed for $k = 2, ..., n$, (and not only for $k = 2, \ldots, s$) and then used in the order $k = n, n-1, ..., 1$, to derive that $c_n, c_{n-2}, ..., c_1$, are 0. Therefore the proof seems to be beyond repair.

(ii) Foregger in examining Eberlein's theorem, claims on p. 385 that the function $\phi = \phi(x,y,z) = xyz - 0.5(xy + xz + yz)$ assumes on $C_{5/4} = \{(x,y,z) : x + y + z = 5/4\} \cap ([3/8, 5/8] \times [3/8, 5/8] \times [1/8, 3/8])$ in $p_0 = (1/2, 1/2, 1/4)$ an interior maximum of value $-0.1875 = \phi(p_0)$. It is easily seen that this would contradict his own (and our) main theorem. Indeed numerical experiments indicate that p_0 is not an interior local extremum.

c. In [11], Waterhouse gives examples showing that the symmetric point in general needs not be a local extremum of symmetric functions subject to symmetric conditions. He also explains why, however, in many cases it will be a local extremum. We do not know if under the conditions of our main theorem the case loc.extr$(\phi) = \emptyset$ can actually happen. But Waterhouse's analysis, as well as Keilson's discussion in [5, p. 220], show this will be very rarely the case.

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