REAL ALGEBRAIC GEOMETRY LECTURE NOTES (25: 13/07/15)

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Contents

Review

1. Chapter I: Valued vector spaces

Let us summarize:

Theorem 1.1. *(Hahnsandwiching Theorem)*
Let V be a valued $\mathbb{Q}$-vector space with skeleton $S(V) = [\Gamma, \{B(\gamma) : \gamma \in \Gamma\}]$. Then

$$\coprod_{\Gamma} B(\gamma) \hookrightarrow V \hookrightarrow \mathrm{H}_{\Gamma} B(\gamma).$$

Two big steps:

(1) $\coprod_{\Gamma} B(\gamma) \hookrightarrow V$.

- we developed the notion of $\mathcal{B} \subset (V, v)$ to be a valuation basis.
- we showed the existence of a maximal valuation independent subset $\mathcal{B}_0$ of (V, v) and proved that $(< \mathcal{B}_0 >_{\mathbb{Q}}, v >) \subseteq (V, v)$ is an immediate extension.
- we noted that $\coprod_{\Gamma} B(\gamma)$ admits a valuation basis and that the converse is true, i.e. whenever (V, v) admits a valuation basis, then $(V, v) \cong (\coprod_{\Gamma} B(\gamma), v_{\min})$.
$(S(V) = [\Gamma, \{B(\gamma) : \gamma \in \Gamma\}])$
- so in general we proceeded as follows:
  - Given (V, v), choose some maximal valuation independent subset $\mathcal{B}_0$.

– Set $V_0 =< \mathcal{B}_0 >_{\mathbb{Q}}$. Then V_0 admits a valuation basis, namely $\mathcal{B}_0$.

– $\bigcup_\Gamma B(\gamma) \cong V_0$, so $\bigcup_\Gamma B(\gamma) \hookrightarrow V$.

(2) $V \hookrightarrow \mathrm{H}_\Gamma B(\gamma)$.

- we first showed that maximally valued $\Leftrightarrow$ pseudo complete.
- we showed that $\mathrm{H}_\Gamma B(\gamma)$ is pseudo complete.
- we proved that if $V_1'|V_1$ is immediate and $y \in V_1'\backslash V_1$, then y is a pseudo-limit of a pseudo-Cauchy sequence in V_1 with no pseudo-limit in V_1.

2. Chapter II: Valuations on ordered fields

Theorem 2.1. *(Kaplansky's Sandwich Theorem)*
Let K be a real closed field with $v(K^*) = G$ and $\overline{K} = k$. Then

$$k(G)^{rc} \hookrightarrow K \hookrightarrow k((G)).$$

This was again proved in 2 steps:

(1) We showed $G \hookrightarrow (K^{>0}, \cdot, 1, <)$ and $k \hookrightarrow K$.

(2) We proved the theorem that if k is real closed and G is divisible, then $k((G))$ is real closed. For this we first proved the same theorem with "real closed" replaced with "algebraically closed". Then (Mac Lane) if k is algebraically closed and G is divisible, then $k((G))$ is algebraically closed.

- $k((G))$ is pseudo-complete.
- the value group of an algebraic extension is contained in the divisible hull of the value group.
- the residue field of an algebraic extension is contained in the algebraic closure of the residue field of the original field.

With these results, one can prove that every algebraic extension must be immediate.

3. Chapter III: Convex valuations on ordered fields

We studied the (under inclusion) linearly ordered set of convex valuations in an ordered field, i.e. the rank $\mathcal{R}$ of K. We characterized it via the rank of $v(K^*)$ and the rank of the value set of $v(K^*)$, respectively,

$$K \xrightarrow{v} v(K^*) \xrightarrow{v_G} \Gamma.$$

Theorem 3.1. *(Characterization of valuations compatible with the order $\leqslant$ of K)*
For a valuation w on an ordered field $(K, \leqslant)$, the following are equivalent:

- *w is compatible with $\leqslant$,*
- *K_w is convex,*
- *I_w is convex*
- *$I_w < 1$,*
- *the residue map $K \to Kw$ induces canonically a total order on Kw ($P \mapsto \overline{P}$).*

Moreover, in the addendum, we proved the Baer-Krull Representation Theorem:

$$\{P \in \mathcal{X}(K) : K_v \text{ is } P\text{-convex}\} \xrightarrow{\sim} \mathcal{X}(K_v) \times \{-1, 1\}^I,$$

where $\mathcal{X}(K)$ and $\mathcal{X}(Kv)$ denote the set of all orderings of K and Kv, respectively, and $I := \dim_{\mathbb{F}_2} G/2G$.

4. Chapter IV: Ordered exponentials fields

Consider $(K, +, 0, <) \xrightarrow{\sim} (K^{>0}, \cdot, 1, <)$.

Theorem 4.1. *(Main Theorem)*

(*i*) $(K, +, 0, <) = \mathbb{A} \sqcup (\overline{K}, +, 0, <) \bigcup I_v$,

(*ii*) $(K^{>0}, \cdot, 1, <) = \mathbb{B} \sqcup (\overline{K}^{>0}, \cdot, 1, <) \sqcup 1 + I_v$.

Recall that $\exp_L : \mathbb{A} \xrightarrow{\sim} \mathbb{B}$, $\exp_M : (K, +, 0, <) \xrightarrow{\sim} (\overline{K^{>0}}, \cdot, 1, <)$ and $\exp_R : I_v \xrightarrow{\sim} 1 + I_v$, the left, middle and right exponential functions.

Discussion of necessary valuation-theoretic conditions:

Theorem 4.2. *If $(K, +, 0, 1, <)$ admits a v-compatible exponential, then*

(*i*) $\overline{exp} : (\overline{K}, +, 0, 1, <) \to (\overline{K}^{>0}, \cdot, 1, <)$, *so $\overline{K}$ is an exponential field.*

(*ii*) $S(v(K^*)) = [G^{<0} : \{(\overline{K}, +, 0, <)\}]$.

Example 4.3.

- Constructing real closed fields which do not admit an exponential function.
  Countable case: a countable divisible ordered abelian group (non-Archimedean) is an exponential group $\Leftrightarrow \cong \bigcup_{\mathbb{Q}} A$, where A is a countable Archimedean divisible ordered abelian group.
- exp is defined on I_v by Neumann's lemma, $\exp(\varepsilon) = \sum \frac{\varepsilon^i}{i!}$. So $\mathbb{K} = k((G))$ always admit $\exp_R$.

Theorem 4.4. $\mathbb{K}$ *never admits an* $\exp_L$.

Question: Does every real closed field admit $\exp_R$?

- True for countable fields.
- True for fields of power series.
- Otherwise?