REAL ALGEBRAIC GEOMETRY LECTURE NOTES (14: 01/06/15 - CORRECTED ON 03/06/19)

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Contents

1. Hardy fields

Today we want to define the canonical valuation on a Hardy field H. For this purpose we observe:

Remark 1.1. (Monotonicity of germs)
Let H be a Hardy field and $f \in H$, $f' \neq 0$. Since $f' \in H$ is ultimately strictly positive or negative, it follows that f is ultimately strictly increasing or decreasing. Therefore

$$\lim_{x \to +\infty} f(x) \in \mathbb{R} \cup \{-\infty, \infty\}$$

exists.

Example 1.2.

(i) $\mathbb{R}$ and $\mathbb{Q}$ are Archimedean Hardy fields (constant germs)

(ii) Consider the set of germs of real rational functions with coefficients in $\mathbb{R}$ (multivariate). By abuse of denation denote it by $\mathbb{R}(X)$. Verify that this is a Hardy field.
Note that with respect to the order defined on a Hardy field, this is a non-Archimedean field, because the function X is ultimately $> N$ for all $N \in \mathbb{N}$.

2. The natural valuation of a Hardy field

Definition 2.1. (The canonical valuation on a Hardy field H).
Let H be a Hardy field. Define for $0 \neq f, g \in H$

$$f \sim g \;\Leftrightarrow\; \lim_{x \to \infty} \frac{f(x)}{g(x)} = r \in \mathbb{R} \setminus \{0\}.$$

This is an equivalence relation, called **asymptotic equivalence relation**. Denote the equivalence class of $0 \neq f$ by $v(f)$. Define

$$v(0) := \infty \text{ and } v(f) + v(g) := v(fg),$$

Moreover, define an order on the set $\{v(f) : f \in H\}$ by setting

$$\infty = v(0) > v(f) \text{ for } f \neq 0.$$

and

$$v(f) > v(g) \ \Leftrightarrow\ \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

Verify that $(v(H), +, <)$ is a totally ordered abelian group.

Lemma 2.2. *The map*

$$\begin{array}{rcl} v : H & \longrightarrow & v(H) \cup \{\infty\} \\ 0 \neq f & \mapsto & v(f) \\ 0 & \mapsto & \infty \end{array}$$

is a valuation and it is equivalent to the natural valuation.

Remark 2.3.

$$R_v = \{f : \lim_{x\to\infty} f(x) \in \mathbb{R}\}.$$

$$I_v = \{f : \lim_{x\to\infty} f(x) = 0\}.$$

$$\mathcal{U}_v = \{f : \lim_{x\to\infty} f(x) \in \mathbb{R}^\times\}.$$

3. Construction of non-Archimedean real closed fields

Our next goal is to prove the following:

Theorem 3.1. *(Main Theorem of chapter 2)*
Let $k \subseteq \mathbb{R}$ be a subfield, G a totally ordered abelian group and $\mathbb{K} := k((G))$. Then $\mathbb{K}$ is a real closed field if and only if

(i) G is divisible,

(ii) k is a real closed field.

Remark 3.2. Once the Main Theorem is proved we can proceed as follows (starting from $\mathbb{R}$) to construct non-Archimedean real closed fields:

(1) Let $\emptyset \neq \Gamma$ be a totally ordered set.

(2) Choose divisible subgroups of $(\mathbb{R}, +, 0, <)$, say $\{B_\gamma : \gamma \in \Gamma\}$ (note that $\mathbb{R}$ is a $\mathbb{Q}$-vector space).

(3) Take $\coprod_{\gamma\in\Gamma} B_\gamma \subset G \subset \mathrm{H}_{\gamma\in\Gamma} B_\gamma$. Note that G is a divisible ordered abelian group.

(4) Take $k \subset \mathbb{R}$ a subfield and consider $k^{\mathrm{rc}} = \{\alpha \in \mathbb{R} : \alpha \text{ alg. over } k\}$. Then $k^{\mathrm{rc}} \subset \mathbb{R}$ is a real closed field (because $\mathbb{R}$ is real closed).

(5) Set $\mathbb{K} = k^{\mathrm{rc}}((G))$.

In the next chapters, we will show "Kaplansky's embedding theorem": any real closed field is a subfield of such a $\mathbb{K}$.

4. Towards the proof of the Main Theorem

Let $k \subset \mathbb{R}$ and G be an ordered abelian group.

Proposition 4.1. *Set $\mathbb{K} = k((G))$ and $v = v_{\min}$. If $\mathbb{K}$ is real closed, then G is divisible and k is a real closed field.*

Proof. We first prove that G is divisible. So let $g \in G$ and $n \in \mathbb{N}$. We have to show that $\frac{g}{n} \in G$. Assume without loss of generality $g > 0$. Consider $\mathbb{K} \ni s = t^g > 0$ in the lex order on $\mathbb{K}$.

(Note that a real closed field R is "root closed for positive elements": For some $s > 0$ consider $x^n - s$. Then $0^n - s < 0$ and $(s+1)^n - s > 0$. The Intermediate Value Theorem gives a root in the interval $]0, s+1[$).

Since $\mathbb{K}$ is real closed take $y = \sqrt[n]{s} \in \mathbb{K}$. Then $v(s) = g$ and thus $v(y) = \frac{g}{n} \in G$.

To show that k is a real closed field let $n \in \mathbb{N}$ be odd and consider some polynomial
$$x^n + c_{n-1}x^{n-1} + \ldots + c_0 \in k[X] \subseteq \mathbb{K}[X].$$
Since $\mathbb{K}$ is real closed, we find some $x \in \mathbb{K}$ such that x is a root of this polynomial, i.e.
$$x^n + c_{n-1}x^{n-1} + \ldots + c_0 = 0.$$
Note that the residue field of $\mathbb{K}$ is k and the residue map is a homomorphism. We want to compute $\overline{c}$ for $c \in k$. Note that $s = c = ct^0 \in k$ so $v_{\min}(c) = 0$ and $\overline{c} = c$. So the residue map is just the identity on k. It remains to show that $v(x) \geqslant 0$. Assume $v(x) < 0$. Then
$$v(x^n + \ldots + c_0) = v(0) = \infty,$$
a contradiction.

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