Polynomial Optimization Efficiency Moments Algebra

POEMA

* Final workshop *

~ The generalised truncated ~
~ Moment Problem ~

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_ Paris _
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Report on the paper:

The truncated moment problem for unital commutative $\mathbb{R}$-algebras

joint work with

R. Curto, M. Ghasemi, M. Infusino

to apper in J. Operator Theory

- 50 pages -

Dedicated to the memory of Murray Marshall.

In a nutshell:

1a. The finite-dimensional K-B-TMP on A:

$A = \mathbb{R}[x_1, \dots, x_n] := \mathbb{R}[\underline{x}]$, $\underset{\text{closed}}{K \subseteq} \mathbb{R}^n$; $B = \mathbb{R}[\underline{x}]_d \leq \mathbb{R}[\underline{x}]$ some $d \in \mathbb{N}$

R. Curto–L. Fialkow, ...

1b. The finite-dimensional K-$\langle \mathcal{A} \rangle$-TMP on A:

$A = \mathbb{R}[\underline{x}]$, $\underset{\text{closed}}{K \subseteq} \mathbb{R}^n$, $\mathcal{A}$ a finite set of monomials, $B = \langle \mathcal{A} \rangle \leq \mathbb{R}[\underline{x}]$

J. Nie, ...

2. The generalised K-B-TMP on A:

A a commutative unital $\mathbb{R}$-algebra

$K \underset{\text{closed}}{\subseteq} \mathcal{X}(A) :=$ the character space of A

$B \leq A$ a linear subspace.

3. Three main theorems:

Theorem I: A is endowed with a seminorm.

Theorem II:

A arbitrary

K compact

Conditions on B

Theorem III:

A arbitrary

K arbitrary

more conditions on B

4. Applications:

(i) The finite dimensional K-TMP

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5. Other applications:

(ii) The TMP for point processes.

(iii) Triangular, Rectangular, sparse TMP.

(iv) Connections with the subnormal completion problem.

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1. The classical TMP: $A = \mathbb{R}[\underline{x}]$, $K \subseteq \mathbb{R}^n$ closed, $B \leq A$

- $L : B \to \mathbb{R}$ a linear functional is said to be K-positive if $L(p) \geq 0$ for all $p \in B$ which are nonnegative on K.
- For a set of monomials $\mathcal{A} \subseteq \mathbb{R}[\underline{x}]$ and a K-positive linear functional L on $\langle \mathcal{A} \rangle$, the $\mathcal{A}$-truncated K-moment problem is the question of establishing whether such L can be represented as an integral w.r.t a positive Radon measure whose support is contained in K.
- When $\mathcal{A} = \{ \underline{x}^\alpha \, ; \, \alpha \in \mathbb{N}_0^n, \, |\alpha| \leq d \}$ some $d \in \mathbb{N}$, the $\mathcal{A}$-TMP is known as the classical TMP.

Answers by Curto–Fialkow (and others):

Theorem (K – compact)

If K is compact and $\langle \mathcal{A} \rangle$ contains a polynomial strictly positive on K, then any K-positive linear functional L defined on $\langle \mathcal{A} \rangle$ is K-represented by a measure.

Theorem ($\mathbb{R}[\underline{x}]_{2d}$ or $\mathbb{R}[\underline{x}]_{2d+1}$)

If $B = \mathbb{R}[\underline{x}]_{2d}$ or $\mathbb{R}[\underline{x}]_{2d+1}$, then a K-positive linear functional L defined on B is K-represented by a measure iff L admits a K-positive extension to $\mathbb{R}[\underline{x}]_{2d+2}$.

2. The general TMP.

- A commutative unital $\mathbb{R}$-algebra
- $\mathcal{X}(A)$: the set of all real valued $\mathbb{R}$-algebra homomorphisms on A.
- $\mathcal{X}(A) \subseteq \mathbb{R}^A$
- For $a \in A$, $\hat{a} : \mathcal{X}(A) \longrightarrow \mathbb{R}$ defined by $\hat{a}(\alpha) := \alpha(a)$
- $\mathcal{X}(A)$ is endowed with the weakest topology which makes all maps $\hat{a}$ continuous.
- this topology coincide with the subspace topology on $\mathcal{X}(A)$ inherited from the product topology on $\mathbb{R}^A$
- we always shall assume that $\mathcal{X}(A)$ is nonempty.

- For any subset $S \subseteq A$ define

$$K_S := \{\alpha \in X(A) : \alpha(S) \subseteq [0,\infty)\}.$$

- <u>Example (finite dimension).</u>

$A = \mathbb{R}[\underline{x}]$, $X(A) \simeq \mathbb{R}^n$, $\hat{p} = p \quad \forall p \in A$

$K_S = \{x \in \mathbb{R}^n : q(x) \geq 0 \quad \forall q \in S\}.$

<u>Definition:</u> Let A be a unital commutative $\mathbb{R}$-algebra, $K \subseteq X(A)$ closed, $B \leq A$ a linear subspace, $L : B \to \mathbb{R}$ a linear functional.

The B-K-truncated moment problem asks whether there exists a positive Radon measure ν whose support is contained in K such that

$$L(b) = \int \hat{b} \, d\nu \qquad \text{for all } b \in B.$$

3a. The case of a seminormed algebra.

Definitions • a <u>submultiplicative seminorm on A</u>

$\rho : A \longrightarrow [0,\infty)$ s.t

$\forall a \in A, r \in \mathbb{R}: \rho(ra) = |r|\rho(a)$

$\forall a, b \in A \qquad \rho(a+b) \leq \rho(a) + \rho(b)$

$\forall a, b \in A \qquad \rho(ab) \leq \rho(a)\rho(b)$.

• <u>Gelfand Spectrum</u>:

$$Sp_\rho(A) = \{\alpha \in \mathcal{X}(A) : |\alpha(a)| \leq \rho(a) \ \forall a \in A\}$$

• $Sp_\rho(A)$ is compact.

• Let $C \subseteq A$ be a cone, for $a \in A$ define

$$\|a\|_{C,\rho} := \inf_{p \in C} \rho(a+p)$$

Theorem I. Let (A, ρ) be a seminormed algebra $B \subseteq A$ linear subspace, S a quadratic module in A and $L : B \to \mathbb{R}$ a linear functional.
Then L admits an integral representation w.r.t a positive Radon measure (whose support is contained in $sp_\rho(A) \cap K_S$) iff $\exists D > 0$ s.t

$$|L(b)| \leq D \|b\|_{S,\rho} \quad \forall b \in B.$$

proof is based on corollary 3.8 of
Ghasemi, Kuhlmann, Marshall
Applications of Jacobi's Representation Theorem to lmc topological algebras, in J. Funct. Anal. 266 (2014)

3b. Theorem II (Compact case).

Let A be a commutative unital $\mathbb{R}$-algebra, $K \subseteq X(A)$ compact, $B \leq A$ a linear subspace such that there exists $q \in B$ with $\hat{q}$ strictly positive on K.

Then every K-positive linear functional $L: B \longrightarrow \mathbb{R}$ admits an integral representation by a positive Radon measure supported in K.

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proof applies Theorem I with

- $A = C(K) =$ algebra of continuous real valued functions on the topological space K equipped with
- $\rho_K(f) := \sup_{\alpha \in K} |f(\alpha)| \quad \forall f \in C(K)$
- $\bar{L}$ a bounded extension of L (using Choquet's lemma). ∎

3c: **Theorem III.** Let A be a commutative unital $\mathbb{R}$-algebra, $K \subseteq \mathcal{X}(A)$ closed, $B \subseteq A$ a linear subspace such that $\exists\, p \in A \setminus B$ with $\hat{p} \geq 1$ on K, $B_p := \langle B, p \rangle \ni 1$, B_p generates A, and

$$\sup_{\alpha \in K} \left| \frac{\hat{b}(\alpha)}{\hat{p}(\alpha)} \right| < \infty \qquad \forall\, b \in B.$$

Let $L : B \to \mathbb{R}$ be a K-positive linear functional. If L has a K-positive extension to B_p, then there exists a K-representing measure for L

i.e. $\qquad L(b) = \int \hat{b}\, d\nu \qquad \forall\, b \in B.$ ∎

4. Applications to the Classical TMP.

Lemma. For any monomial $\underline{x}^\alpha$ of degree $2d$ or $2d+1$ there exists a polynomial $p \in \mathbb{R}[\underline{x}]_{2d+2}$ such that $|\underline{x}^\alpha| \leq p(\underline{x})$ and $p \geq 1$ on $\mathbb{R}^n$.

proof: write $\alpha = \gamma + 2\beta$ with
$\gamma = (\gamma_1, \dots, \gamma_n)$, $\gamma_i \in \{0,1\}$ $\forall i = 1, \dots, n$
so $\underline{x}^\alpha = \underline{x}^\gamma \underline{x}^{2\beta}$.

- if $\gamma = (0,\dots,0)$ set $p := \prod_{i=1}^{n} (1 + x_i^2)^{\beta_i}$
- if $\gamma \neq (0,\dots,0)$ set

$$p = \frac{1}{|\gamma|}\left(\sum_{i=1}^{n} \gamma_i (1+x_i^2)^{\left\lceil \frac{|\gamma|+1}{2} \right\rceil}\right) \prod_{i=1}^{n} (1+x_i^2)^{\beta_i}.$$

(use AG inequality). ∎

Corollary. Let $\mathcal{P} \subseteq \mathbb{R}[\underline{x}]_k$. Then $\exists\, p \in \mathbb{R}[\underline{x}]_{k+1}$ if k is odd and $p \in \mathbb{R}[\underline{x}]_{k+2}$ if k is even such that $p \geq 1$ and
$$\sup_{\underline{y} \in \mathbb{R}^n} \left| \frac{f(\underline{y})}{p(\underline{y})} \right| < \infty$$
for all $f \in \mathcal{P}$. $\blacksquare$

Corollary to Th II: Let $K \subseteq \mathbb{R}^n$ closed, L a K-positive linear functional on $\mathbb{R}[\underline{x}]_{2d}$ (respectively $\mathbb{R}[\underline{x}]_{2d+1}$). There exists $p \in \mathbb{R}[\underline{x}]_{2d+2}$ such that $p \geq 1$ and $\sup_{y \in \mathbb{R}^n} \left| \frac{f(y)}{p(y)} \right| < \infty \quad \forall f \in \mathbb{R}[\underline{x}]_{2d}$.
Thus L has a rep. measure iff L admits a

K - positive extension to B_p. $\boxed{\equiv}$

This corollary slightly improves the result of Curto and Fialkow because it requires the K - positive extension to a subspace $B_p \subseteq \mathbb{R}[\underline{x}]_{2d+2}$, instead of requiring it for the whole $\mathbb{R}[\underline{x}]_{2d+2}$.

Corollary to Th II. Let $K \subseteq \mathbb{R}^n$ compact $B \subseteq \mathbb{R}[\underline{x}]_d$ s.t $\exists\, p \in B$, $p > 0$ on K. Let $L : B \longrightarrow \mathbb{R}$ be a K- positive linear functional. Then L admits a representing measure. $\boxed{\equiv}$

Thank

you !