A Tale of Two Cones: PSD vs SOS in equivariant situations

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PSD vs SOS forms

- For $n \in \mathbb{N}$, a polynomial $p(x) \in \mathbb{R}[\underline{x}] = \mathbb{R}[x_1, \ldots, x_n]$ is called
  - **nonnegative or positive semidefinite (psd)** if $p(x) \geq 0 \ \forall x \in \mathbb{R}^n$
  - a **sum of squares (sos)** if $p = \sum_i q_i^2$ for some $q_i \in \mathbb{R}[\underline{x}]$
- Clearly every sos is psd.
- Converse: When can a psd polynomial written as a sos of poly's?
  - Sufficient to consider this question for **forms** (i.e. **homogeneous polynomials**) of even degree.
- Let $\mathcal{F}_{n,2d}$ be the $\binom{n+2d-1}{n-1}$-dimensional vector space of all real forms in n variables and degree $2d$, called **n-ary 2d-ics**, where $n, d \in \mathbb{N}$.
- $\mathcal{P}_{n,2d} := \{f \in \mathcal{F}_{n,2d} \mid f \text{ is psd } \}$, the set of psd forms.
- $\Sigma_{n,2d} := \{f \in \mathcal{F}_{n,2d} \mid f \text{ is sos}\}$, the set of sos forms.

PSD vs SOS forms

Theorem (Hilbert, 1888):
$\mathcal{P}_{n,2d} = \Sigma_{n,2d}$ if and only if $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$.

- Hilbert proved that $\Sigma_{3,6} \subsetneq \mathcal{P}_{3,6}$ and $\Sigma_{4,4} \subsetneq \mathcal{P}_{4,4}$, and demonstrated that it is enough for all remaining cases, i.e.

  Proposition [Reduction to Basic cases]:
  If $\Sigma_{3,6} \subsetneq \mathcal{P}_{3,6}$ and $\Sigma_{4,4} \subsetneq \mathcal{P}_{4,4}$, **then**
  $\Sigma_{n,2d} \subsetneq \mathcal{P}_{n,2d}$ for all $n \geq 3, 2d \geq 4$ and $(n, 2d) \neq (3, 4)$.

- **(Motzkin, 1967)**

  $M := z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2 \in \mathcal{P}_{3,6} \setminus \Sigma_{3,6}$.

- **(Robinson, 1969)**

  $$W := x^2(x-w)^2 + \big(y(y-w) - z(z-w)\big)^2 \\ +2yz(x+y-w)(x+z-w) \in \mathcal{P}_{4,4} \setminus \Sigma_{4,4}$$

PSD vs SOS forms invariant under the action of S_n

- A form $f \in \mathcal{F}_{n,2d}$ is called **symmetric** if $\forall\ \sigma \in S_n$:
  $\sigma f(x_1, \ldots, x_n) := f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ is equal to $f(x_1, \ldots, x_n)$.
- Let $S\mathcal{P}_{n,2d}$ and $S\Sigma_{n,2d}$ be the cones of n-ary 2d-ic symmetric forms which are psd and sos respectively.
- **Theorem (Choi-Lam, 1976; G.-Kuhlmann-Reznick, 2015):**
  $S\mathcal{P}_{n,2d} = S\Sigma_{n,2d}$ iff $n = 2$ or $2d = 2$ or $(n, 2d) = (3, 4)$.

  Proposition [Reduction to Basic cases]:
  If $S\Sigma_{3,6} \subsetneq S\mathcal{P}_{3,6}$ and $S\Sigma_{n,4} \subsetneq S\mathcal{P}_{n,4}\ \forall\ n \geq 4$, **then**
  $S\Sigma_{n,2d} \subsetneq S\mathcal{P}_{n,2d}$ for all $n \geq 3, 2d \geq 4$ and $(n, 2d) \neq (3, 4)$.
- (Robinson, 1969)
  $$R := x^6 + y^6 + z^6 + 3x^2y^2z^2 - (x^4y^2 + y^4z^2 + z^4x^2 + x^2y^4 + y^2z^4 + z^2x^4) \in S\mathcal{P}_{3,6} \setminus S\Sigma_{3,6}.$$
- (Choi-Lam, 1976)
  $$f_{4,4} := \sum^6 x^2y^2 + \sum^{12} x^2yz - 2xyzw \in S\mathcal{P}_{4,4} \setminus S\Sigma_{4,4}.$$
- (G.-Kuhlmann-Reznick, 2015)
  $F_{n,4} \in S\mathcal{P}_{n,4} \setminus S\Sigma_{n,4}$ for $n \geq 5$.

PSD vs SOS forms invariant under the action of $S_n \times \mathbb{Z}_2^n$

- A form $f \in \mathcal{F}_{n,2d}$ is called **even symmetric** if it is symmetric and in each term of f every variable has even degree.
- Let $S\mathcal{P}^e_{n,2d}$ and $S\Sigma^e_{n,2d}$ are cones of n-ary 2d-ic even symmetric forms which are psd and sos respectively.
- **Theorem (G.-Kuhlmann-Reznick, 2016):**
  $S\mathcal{P}^e_{n,2d} = S\Sigma^e_{n,2d}$ iff $n = 2$ or $d = 1$ or $(n, 2d) = (n, 4)_{n\geq 3}$ or $(3, 8)$.

  Proposition [Reduction to Basic cases]:
  If $S\Sigma^e_{n,2d} \subsetneq S\mathcal{P}^e_{n,2d}$ for $(n, 6)_{n\geq 3}, (n, 8)_{n\geq 4}, (n, 10)_{n\geq 3}$, $(n, 12)_{n\geq 3}$,
  then $S\Sigma^e_{n,2d} \subsetneq S\mathcal{P}^e_{n,2d}$ for all $n \geq 3, 2d \geq 6$ and $(n, 2d) \neq (3, 8)$.
- (Choi-Lam-Reznick, 1987)
  $F_{n,6} \in S\mathcal{P}^e_{n,6} \setminus S\Sigma^e_{n,6}$ for $n \geq 3$.
- (Harris, 1999)
  $F_{n,2d} \in S\mathcal{P}^e_{n,2d} \setminus S\Sigma^e_{n,2d}$ for $(n, 2d) = (3, 10), (4, 8)$.
- (G.-Kuhlmann-Reznick, 2016)
  $F_{n,2d} \in S\mathcal{P}^e_{n,2d} \setminus S\Sigma^e_{n,2d}$ for $(n, 8)_{n\geq 4}, (n, 10)_{n\geq 3}$ and $(n, 12)_{n\geq 3}$.

Finite Reflection Groups

- **Goal: Hilbert's 1888 theorem for psd and sos forms invariant under the action of a finite reflection group**
- Let V be a finite Euclidean space endowed with a positive definite symmetric bilinear form. Given a non-zero vector $\alpha \in V$, we define the linear operator s_α by $s_\alpha(\lambda) := \lambda - \frac{2<\lambda,\alpha>}{<\alpha,\alpha>}\alpha$ for any $\lambda \in V$.
- s_α is an orthogonal transformation, i.e. $< s_\alpha(\lambda), s_\alpha(\mu) >=< \lambda, \mu >$ for all $\lambda, \mu \in V$.
- $s_\alpha^2 = 1$, i.e. s_α is an element of order 2 of the group $O(V)$ of all orthogonal transformations of V.
- A finite subgroup of $O(V)$ generated by reflections is called a finite reflection group[1].

[1](Reflection groups and Coxeter groups, J.E. Humphreys, page-3)

Finite Reflection Groups

For $s \in G$, where G is a finite reflection group.

- If s is the linear operator on $\mathbb{R}^n$ then the corresponding action on $\mathbb{R}[x]$ is defined as:

  $$\text{for } f \in \mathcal{F}_{n,2d},\ sf(x_1, \ldots, x_n) := f(s(x_1, \ldots, x_n)) \in \mathcal{F}_{n,2d}.$$

  In particular if $G = S_n$, G acts on $\mathbb{R}^n$ by permuting the coordinates of a given n tuple of reals, so defining the corresponding action on $\mathbb{R}[x]$ gives

  $sf(x_1, \ldots, x_n) = f(s(x_1, \ldots, x_n)) = f(x_{s(1)}, \ldots, x_{s(n)}) \in \mathcal{F}_{n,2d}$.

- If the linear operator s on $\mathbb{R}^n$ is represented w.r.t. the standard basis by the $n \times n$ matrix A_s, then the action description becomes:

  $$\text{for } f \in \mathcal{F}_{n,2d},\ sf(x_1, \ldots, x_n) := f(A_s \begin{bmatrix} x_1 \\ \ldots \\ x_n \end{bmatrix}) \in \mathcal{F}_{n,2d}.$$

Finite Reflection Groups

- Let G be a finite group that acts linearly on $\mathbb{R}[x]$. Denote by

  $$\mathbb{R}[x]^G := \{f \in \mathbb{R}[x] \mid \sigma.f := f \ \forall \ \sigma \in G\}$$

  the subspace of G-invariant polynomials.

- For a group G, denote by $\Sigma^G_{n,2d}$ and $\mathcal{P}^G_{n,2d}$ respectively the cones of n-ary 2d-ic forms invariant under G which are psd and sos.

- When a reflection group G acts on $V = \mathbb{R}^n$ with no nonzero fixed points, we say that G is **essential** relative to V.

- Any real reflection group can be identified with a direct product of essential reflection groups.

Finite Reflection Groups

According to Coxeter classification, the real reflection groups have been classified and are precisely:

- the four infinite families of essential reflection groups:
  - $\mathbf{A}_{\boldsymbol{n}}(\boldsymbol{n} \geq \mathbf{1})$ [identified with Symmetric S_{n+1}],
  - $\mathbf{B}_{\boldsymbol{n}}(\boldsymbol{n} \geq \mathbf{2})$ [identified with $S_n \times \mathbb{Z}_2^n$],
  - $\mathbf{D}_{\boldsymbol{n}}(\boldsymbol{n} \geq \mathbf{4})$ [Subgroup of index 2 in the group of type B_n], and
  - $\mathbf{I}_{\mathbf{2}}(\boldsymbol{m})(\boldsymbol{m} \geq \mathbf{3})$ [Dihedral group of order $2m$ acting on the euclidean plane]
- the six exceptional reflection groups $\mathbf{E_6}$, $\mathbf{E_7}$, $\mathbf{E_8}$, $\mathbf{F_4}$, $\mathbf{H_3}$, $\mathbf{H_4}$.

PSD vs SOS forms invariant under the action of a finite reflection group

- The dihedral group of order 2m, denoted by $I_2(m)$, is the symmetry group of the regular m-gon and is a finite reflection group.
  - **Proposition:**
    $I_2(m)$-invariant forms are psd if and only if they are sos, i.e.,
    $\Sigma_{2,2d}^{I_2(m)} = \mathcal{P}_{2,2d}^{I_2(m)}$ for all n and d.

    Proof. $I_2(m)$-invariant forms are bivariate. Thus, by Hilbert's 1888 characterisation, these forms are psd if and only if they are sos.
- The signed symmetric group, denoted by B_n, can be identified with $S_n \times \mathbb{Z}_2^n$. It is generated by the reflections at $\{X_i = \pm X_j\}$ for $1 \leq i \leq j \leq n$ and is a finite reflection group.
  - **Proposition:**
    $\Sigma_{n,2d}^{B_n} = \mathcal{P}_{n,2d}^{B_n}$ iff $n = 2$ or $d = 1$ or $(n, 2d) = (n, 4)_{n \geq 3}$ or $(3, 8)$.

    Proof. Forms invariant under B_n corresponds to even symmetric forms.

PSD vs SOS forms invariant under the action of a finite reflection group

- D_n can be identified with $S_n \times \mathbb{Z}_2^{n-1}$. It is the subgroup of B_n of index 2, generated by the reflections at $\{X_i = \pm X_j\}$ for $1 \leq i \leq j \leq n$ with even no. of sign changes.
  - **Theorem (Debus-Riener):**

    $\Sigma_{n,2d}^{D_n} = \mathcal{P}_{n,2d}^{D_n}$ iff $n = 2$ or $d = 1$ or $(n, 2d) = (n, 4)_{n \geq 3}$ or $(3, 8)$.

    Proof. Since $f \in \mathcal{P}_{n,2d}^{B_n} \setminus \Sigma_{n,2d}^{B_n} \Rightarrow f \in \mathcal{P}_{n,2d}^{D_n} \setminus \Sigma_{n,2d}^{D_n}$, even symmetric psd not sos examples work.

    For proving equality in $(4, 4)$ case, they used the following result: $\Sigma_{n,2d}^{G} = \mathcal{P}_{n,2d}^{G}$ *if and only if any extremal ray in the dual cone of* $\Sigma_{n,2d}^{G}$ *is generated by a point-evaluation.*

PSD vs SOS forms invariant under the action of a finite reflection group

- A_{n-1} ($n \geq 2$) can be identified with the symmetric group S_n acting on an $(n-1)$-dimensional euclidean space as a group generated by reflections, fixing no point except the origin.
  - **Work in progress (Debus, G., Kuhlmann, Riener):**

    For all $n \in \mathbb{N}$, $\Sigma_{n,4}^{A_n} = \mathcal{P}_{n,4}^{A_n}$.
  - **Ongoing work:**

    Complete the characterisation for forms invariant under the action of A_n.

PSD vs SOS forms invariant under the action of a finite reflection group

- Consider forms invariant under products of the type $I_2(m)$:
  - **Theorem (Debus, 2019):**
    $\mathcal{P}_{4,4}^{I_2(4)\times I_2(4)} = \Sigma_{4,4}^{I_2(4)\times I_2(4)}$.
  - **Proposition (Debus, G., Kuhlmann, Riener):**
    $\mathcal{P}_{4,4}^{I_2(2)\times I_2(2)} \supsetneq \Sigma_{4,4}^{I_2(2)\times I_2(2)}$.
    Proof. Choi-Lam-Reznick's psd symmetric quaternary quartic which is not sos.

- **Ongoing work:** Consider forms invariant under
  - all products of factors of types A_n, B_n, D_n, $I_2(m)$
  - the exceptional reflection groups E_6, E_7, E_8, F_4, H_3, H_4
  - all products from $A_n, B_n, D_n, I_2(m)$ and exceptional reflection groups

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Thank You