Lexicographic Exponentiation of chains*

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In memory of Felix Hausdorff.

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PLAN OF THE TALK:

Part I: Historical. Hausdorff's 1908 paper.

Part II: Introductory.

- Arithmetic operations: finite sums and products. General lexicographic products and powers.
- Anti-lexicographic products. Proposition. Warning.
- Relation to Ordinal Arithmetic.
- Dependence on the chosen base points. Brief discussion. See [Gr] for more.
- Examples.

Part III: Focus on Results of [HKM] and [K].

(a) 2-transitivity: *when is Aut* (Δ^Γ) *2-transitive ?*

- Hausdorff's interest. Definitions and field example. 2-transitive implies n-transitive.
- General Proposition. Hausdorff's theorem proved in [W]. Converse proved today.
- State main result of [HKM].

(b) Isomorphism Invariants: *Does* $\mathbb{R}^\Gamma \simeq \mathbb{R}^{\Gamma'}$ *imply* $\Gamma \simeq \Gamma'$ *?*

- State main result of [K]. State main result of [HKM]. Converse to Hausdorff's theorem [W].
- Two powerful tools: C_{00} chains and Arithmetic Rules.
- Proofs and examples.

Part IV: Algebraic motivation and applications in [K-K-S1], [K-K-S2] and [K–S].

(*if time permits*).

PART I.[1]

In [H1][2] Hausdorff:

- Introduces operations on chains: sums, products, lexicographic products, lexicographic exponentiation of chains.
- Develops the basic arithmetic of these operations.
- Generalizes Cantor's ordinal arithmetic.
- Uses lexicographic chain constructions for constructing models with given species and genera.
- Formulates the GCH and defines inaccessible cardinals.

He resumes this study in [H2] and investigates (among other problems):

- 2-transitivity of lexicographic products.

The theory offers a variety of open problems that I have been studying in the last decade. Some have been solved, many are still open.

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[1] *Througout this talk, chain means totally ordered set.*

[2] A translation into English of this paper appears as an appendix in [Gr].

PART II.

Arithmetic operations on chains.

Let Γ and Γ' be chains.

The **sum** $\Gamma + \Gamma'$ is the chain formed by concatenation, with $\Gamma < \Gamma'$.

- Note that our definition coincides with ordinal addition in case Γ and Γ' are ordinals.

More generally, if $\{\Gamma_i; i \in I\}$ is a collection of chains indexed by a chain I, we define the sum $\Sigma_{i\in I}\Gamma_i$ analogously.

We denote by $\Gamma \,\vec{\amalg}\, \Gamma'$ the **lexicographic product** of Γ and Γ'. That is, $\Gamma \,\vec{\amalg}\, \Gamma'$ is the chain obtained by ordering the Cartesian product $\Gamma\times\Gamma'$ lexicographically from the left. Note that

- $\Gamma \,\vec{\amalg}\, \Gamma' \simeq \Sigma_{\gamma\in\Gamma}\Gamma'$ (see figure)
- if α and β are ordinals then $\alpha \,\vec{\amalg}\, \beta$ is the ordinal product $\beta\alpha$ (**!**)

Lexicographic exponentiation of chains:

Now let $\{\Delta_\gamma\,;\,\gamma\in\Gamma\}$ be chains, with index chain Γ, and for each $\gamma\in\Gamma$, fix a base point $0_\gamma\in\Delta_\gamma$.

We define a chain in the Cartesian product $\Pi_{\gamma\in\Gamma}\Delta_\gamma$: the **lexicographic product** is the subset

$$\mathop{\mathbf{H}}_{\gamma\in\Gamma}\Delta_\gamma := \{s\in\prod_{\gamma\in\Gamma}\Delta_\gamma\,;\ \text{support}\,(s)\ \text{is wellordered}\},$$

totally ordered lexicographically from the left (also known as "order by first differences"). Here, support $(s) := \{\gamma\in\Gamma\,;\ s(\gamma)\neq 0_\gamma\}$.

If all Δ_γ's are the same chain Δ, and all base points 0_γ are the same element $0\in\Delta$, then $\mathbf{H}_{\gamma\in\Gamma}\Delta_\gamma$ is the **lexicographic power** Δ^Γ:

$$\begin{aligned}\Delta^\Gamma &:= \{s:\Gamma\to\Delta\,;\ \text{support}\,(s)\ \text{is wellordered}\}\\ &= \{s\in\prod_{\gamma\in\Gamma}\Delta\,;\ \text{support}\,(s)\ \text{is wellordered}\}.\end{aligned}$$

Dual Theory: Anti-lexicographic exponentiation of chains.

• In the literature, many authors prefer to work with the anti-lexicographic ordering.

The **anti-lexicographic power** ${}^{\Gamma}\Delta$ is the set

$$ {}^{\Gamma}\Delta := \{ s : \Gamma \to \Delta \, ; \, \text{support}(s) \text{ is anti-wellordered in } \Gamma \}, $$

ordered anti-lexicographically from the right (also known as "ordered by last differences").

How to translate from lex to anti-lex ?

Let Γ^* denote Γ with its order reversed. We note:

Proposition 1

Let Γ be a chain, and Δ a chain with a base point 0. Then the anti-lexicographic power ${}^{\Gamma}\Delta$ coincides with the lexicographic power Δ^{Γ^*}.

• But note that in general $\Delta^{\Gamma} \simeq \Delta^{\Gamma'}$ *does not imply* ${}^{\Gamma}\Delta \simeq {}^{\Gamma'}\Delta$. Example later.

Relation to Ordinal Arithmetic.

When α and β are ordinals, our lexicographic power α^{β^*}, *with chosen base point the least element* $0 \in \alpha$, is the ordinal α^{β}.

That is, our anti-lexicographic power ${}^{\beta}\alpha$, *with chosen base point the least element* $0 \in \alpha$, is the ordinal α^{β}.

• In order to recover Cantor's notation, Hausdorff writes Δ^{Γ} whenever he actually works with the lexicographic power Δ^{Γ^*}.

• It is important that here, the chosen base point is the least element $0 \in \alpha$. For example, if α is the ordinal $2 = \{0, 1\}$, then the lexicographic power 2^{β^*}, if computed with base point $1 \in \{0, 1\}$ instead of 0, is the *reverse* of the ordinal 2^{β}.

Dependence on the chosen base points.

The lexicographic chain Δ^Γ depends in general on the choice of the base point 0 of Δ.[3] Below is a brief discussion of this issue.

A uniform way of defining lexicographic products:

In [H1] Hausdorff introduces lexicographic products as follows.

Given $\{\Delta_\gamma\ ;\ \gamma \in \Gamma\}$ with index chain Γ, define a **partial order on the Cartesian product** $\Pi_{\gamma\in\Gamma} \Delta_\gamma$ by comparing two sequences s and t lexicographically from the left just in case

$$\mathrm{dif}(s,t) := \{\gamma \in \Gamma\ ;\ s(\gamma) \neq t(\gamma)\}$$

has a least element.

Now define an **equivalence relation on the Cartesian product** $\Pi_{\gamma\in\Gamma} \Delta_\gamma$:

$$s \sim t \text{ if } \mathrm{dif}(s,t) \text{ is wellordered.}$$

- The equivalence classes are maximal chains in this partial order.

- Furthermore, if $[s]$ denotes the equivalence class of $s \in \Pi_{\gamma\in\Gamma} \Delta_\gamma$, then each $[s]$ is a lexicographic product defined by s, that is, with base points $0_\gamma = s(\gamma) \in \Delta_\gamma$.

- So if $t \sim s$ then the lexicographic product with base points $0_\gamma = t(\gamma)$ coincides with the lexicographic product defined by s, and conversely.

Each equivalence class is possibly a new chain ....

[3] In [Gr], α^{ω^*} is computed, for every possible choice of the base point!

Cases where it does not matter:

• If Γ is wellordered, then there is a single equivalence class, and the lexicographic product of $\{\Delta_\gamma\,;\,\gamma\in\Gamma\}$ with index chain Γ is uniquely determined (independently from the chosen base points). It is just $\Pi_{\gamma\in\Gamma}\Delta_\gamma$ totally ordered lexicographically.

• If $t \not\sim s$, s and t may still define *isomorphic* lexicographic products. This is the case, for example, if each of the Δ_γ 's is a transitive chain.[4]

• In particular, if Δ is a totally ordered Abelian group, then the lexicographic chain Δ^Γ is uniquely determined, *up to isomorphism*, independently of the chosen base point.

[4]for each $\gamma\in\Gamma$ fix an automorphism π_γ of Δ_γ satisfying $\pi_\gamma(s(\gamma)) = t(\gamma)$; the π_γ's induce the required isomorphism in the obvious way. Moreover, this induced isomorphism maps base points to base points.

Examples.

- $\mathbb{Z}^{\mathbb{N}}$ is the order type of the irrationals.
- $\mathbb{N}^{\mathbb{N}}$ (with any base point) is the order type of the non-negative reals.
- $2^{\mathbb{N}}$ (with any base point) is the order type of Cantor's ternary set.
- The underlying order of Hahn groups is a lexicographic product.
- The underlying order of a field of power series $k((G))$ (with coefficients in an ordered field k and exponents in an ordered Abelian group G) is the lexicographic power k^G. Similarly for the rings of power series $k((G^{\geq 0}))$ and $k((G^{<0}))$.
- The underlying order of Conway's "field of surreal numbers" **No** is a lexicographic power.

Lexicographic orderings appear naturally in:
Descriptive Set Theory, Real Algebra, Valuation Theory, Gröbner Bases (monomial orders), ...

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PART III.

(a) 2-transitivity.

Let A be a chain containing more than 2 elements. A is **2-transitive** if given a_1, a_2, b_1, $b_2 \in A$ such that $a_1 < b_1$ and $a_2 < b_2$, there exists an automorphism σ of A such that $\sigma(a_1) = a_2$ and $\sigma(b_1) = b_2$.

• Example: the underlying chain of a totally ordered field F is always 2-transitive.[5]

• If A is 2-transitive then it is n-transitive for all natural numbers $n \geq 2$ (defined analogously).

• In [H2] Hausdorff's major interest in lexicographic powers is in their 2-transitivity:

when is Aut (Δ^Γ) *2-transitive?*

Proposition 2

A lexicographic power Δ^Γ is 2-transitive if Γ is transitive and Δ is 2-transitive.

[5] Given a_1, a_2, b_1, b_2 as above, define $\sigma(a) = (a - a_1)\frac{(b_2 - a_2)}{(b_1 - a_1)} + a_2$.

However, this proposition does not cover the interesting cases, for example, the case of an ordinal exponent or base.

Theorem [H2], [W]

Let α be an additive principal ordinal. Then $\mathbb{R}^\alpha$ is 2-transitive.

(α is **additive principal**[6] if it is an ordinal power of ω, or equivalently, if α is isomorphic to any of its nonempty final segments, i.e., α is self-final).

Today, we shall prove the converse:

Theorem 1 [HKM]

Let α be an ordinal. If $\mathbb{R}^\alpha$ is 2-transitive, then α is additive principal.

The other main result concerning 2-transitivity is:

Theorem 2 [HKM]

Let Δ be a countable ordinal ≥ 2, with its least element 0 as base point. Then $\Delta^\mathbb{R}$ (with its least element deleted) is 2-transitive.

The proof is quite involved, we shall omit it in today's talk.

[6]That is, α is a *monomial* in the **Cantor normal form**.

(b) Isomorphism Invariants.

We studied the question

$$\textit{Does } \mathbb{R}^{\Gamma} \simeq \mathbb{R}^{\Gamma'} \textit{ imply } \Gamma \simeq \Gamma' \textit{ ?}$$

Theorem [K]

Let α be an ordinal, and J a chain in which the chain $\mathbb{R}$ does not embed. Assume that $\mathbb{R}^{\alpha}$ embeds in $\mathbb{R}^{J}$. Then α embeds in J. In particular, if α and β are distinct ordinals, then $\mathbb{R}^{\alpha} \not\simeq \mathbb{R}^{\beta}$.

What about non-wellordered exponents?

Sometimes, Theorem [K] provides a test:

Example 1

$\mathbb{R}^{\mathbb{N}} \not\simeq \mathbb{R}^{\mathbb{Q}}$.

(Indeed, every countable ordinal embeds in $\mathbb{Q}$ ...)

At other times, the test is not informative, and we have to work harder:

Theorem 4 [HKM] [7]

$\mathbb{R}^{\mathbb{R}} \not\simeq \mathbb{R}^{\mathbb{Q}}$.

Theorem 3 [HKM] [8]

Let α be any countably infinite ordinal. Then $\mathbb{R}^{\alpha^*+\alpha} \simeq \mathbb{R}^{\alpha}$.

We now want to provide the main ideas in the proofs of Theorem 1 and Theorem 3 of [HKM]. For this we need ...

[7] The proof is quite involved, we shall omit it in today's talk.

[8] We prove a more general result: Let α and β be ordinals, with α countable and β infinite. Then $\mathbb{R}^{\alpha^*+\beta} \simeq \mathbb{R}^{\beta}$.

ARITHMETIC RULES:

(1) The operations $+$ and $\vec{\amalg}$ are both associative, but in general not commutative.

(2) $(\Gamma + \Gamma')\,\vec{\amalg}\,\Gamma'' \simeq (\Gamma\,\vec{\amalg}\,\Gamma'') + (\Gamma'\,\vec{\amalg}\,\Gamma'')$.

(3) $(\Gamma_1 + \Gamma_2)^* \simeq \Gamma_2^* + \Gamma_1^*$ and $(\Gamma_1\,\vec{\amalg}\,\Gamma_2)^* \simeq \Gamma_1^*\,\vec{\amalg}\,\Gamma_2^*$.

(4) $(\Delta^\Gamma)^* \simeq (\Delta^*)^\Gamma$.

(5) $\Delta^{\Gamma+\Gamma'} \simeq \Delta^\Gamma\,\vec{\amalg}\,\Delta^{\Gamma'}$.

(6) If $\{\Gamma_i; i \in I\}$ is a collection of chains indexed by a chain I, then

$$\Delta^{\sum_{i\in I}\Gamma_i} \simeq \mathop{\mathbf{H}}_{i\in I} \Delta^{\Gamma_i},$$

(where the base point of Δ^{Γ_i} is $\mathbf{0}$, the sequence with empty support).

(7) In particular $\Delta^{\Gamma\,\vec{\amalg}\,\Gamma'} \simeq (\Delta^{\Gamma'})^\Gamma$. (Recall that $\Gamma\,\vec{\amalg}\,\Gamma' \simeq \sum_{\gamma\in\Gamma}\Gamma'$).

(8) $\Delta^{\Gamma_1} \simeq \Delta^{\Gamma_2}$ and $\Delta^{\Gamma_1'} \simeq \Delta^{\Gamma_2'} \Rightarrow \Delta^{\Gamma_1+\Gamma_1'} \simeq \Delta^{\Gamma_2+\Gamma_2'}$.

(9) $\Delta^{\Gamma_1} \simeq \Delta^{\Gamma_2} \Rightarrow \Delta^{\Gamma'\,\vec{\amalg}\,\Gamma_1} \simeq \Delta^{\Gamma'\,\vec{\amalg}\,\Gamma_2}$.

We also need ...

A powerful tool.

A chain A is C_{00} or has **countable coterminalities** if both the cofinality and the coinitiality of A are equal to $\aleph_0$. That is, there is a coterminal (both coinitial and cofinal) subset of A isomorphic to $\mathbb{Z}$.

Our main tool is the following:

Proposition 3

Let A be a 2-transitive C_{00} chain. Then A is isomorphic to any of its convex C_{00} subsets.

To apply this proposition to 2-transitive C_{00} lexicographic powers and their convex subsets, the following easy observations are very useful:

Remark

Let Γ be a chain and $F \neq \emptyset$ a final segment of Γ. Then Δ^F is (isomorphic to) a convex subset of Δ^Γ.

Proposition 4

Let Γ be a chain, and Δ a chain with base point $0 \in \Delta$. Then the lexicographic power Δ^Γ is C_{00} if either Γ has a least element and Δ is C_{00}, or Γ has countable coinitiality and 0 is not an endpoint of Δ.

We can now work with these facts to establish ...

Proposition 5

$\mathbb{R}^{\omega^*+\omega}$ and $\mathbb{R}^{\omega}$ are isomorphic and these chains are 2-transitive.

Proof

$\mathbb{R}^{\omega}$ is convex in $\mathbb{R}^{\omega^*+\omega}$ by the Remark. Both are C_{00} chains by Proposition 4. $\mathbb{R}^{\omega^*+\omega}$ admits an ordered field structure; in fact, it is the underlying chain of the Laurent series field $\mathbb{R}((\mathbb{Z}))$. Therefore, $\mathbb{R}^{\omega^*+\omega}$ is a 2-transitive chain. Now apply Proposition 3.

Example 2

$\mathbb{R}^{\omega^*+\omega} \simeq \mathbb{R}^{\omega}$. However, the corresponding anti-lexicographic powers are not. If they were, then we would have by Proposition 1 that $\mathbb{R}^{\omega^*+\omega} \simeq \mathbb{R}^{\omega^*}$. So $\mathbb{R}^{\omega} \simeq \mathbb{R}^{\omega^*}$. But this is impossible by Theorem [K] since ω does not embed in ω^*.

The proof of **Theorem 3** now basically follows by an induction argument. This theorem is *not true* without the assumption of countability:

Example 3

Let κ be an uncountable regular cardinal. Then $\mathbb{R}^{\kappa^*+\kappa}$ and $\mathbb{R}^{\kappa}$ are *not* isomorphic. Indeed, $\mathbb{R}^{\kappa}$ is C_{00} by Proposition 4, whereas $\mathbb{R}^{\kappa^*+\kappa}$ has cofinality κ.

We can now also establish **Theorem 1**:

Proof: Assume that $\alpha \neq 1$. Let φ be a nonempty final segment of α. Both $\mathbb{R}^\alpha$ and $\mathbb{R}^\varphi$ are C_{00} by Proposition 4, and $\mathbb{R}^\varphi$ is convex by the Remark. So by Proposition 3, $\mathbb{R}^\alpha \simeq \mathbb{R}^\varphi$. So $\alpha \simeq \varphi$ by Theorem [K]. This shows that α is self-final, as required. $\square$

An example of a slightly different flavour:

Example 4

$\mathbb{R}^\mathbb{R}$ and $\mathbb{R}^{\mathbb{R}+\mathbb{R}}$ are isomorphic and these chains are 2-transitive.

Proof:

$\mathbb{R}^{\mathbb{R}+\mathbb{R}} \simeq \mathbb{R}^\mathbb{R} \vec{\amalg} \mathbb{R}^\mathbb{R}$ (by AR(6)).

Also, $\mathbb{R}^\mathbb{R} \simeq \mathbb{R}^{\mathbb{R}^{<0}}$ (since the exponents are isomorphic).

Further, $\mathbb{R}^\mathbb{R} \simeq \mathbb{R}^{\mathbb{R}^{\geq 0}}$:

$\mathbb{R}^{\mathbb{R}^{\geq 0}}$ is convex in $\mathbb{R}^\mathbb{R}$ by the Remark. Both are C_{00} chains by Prop. 4.

$\mathbb{R}^\mathbb{R}$ is 2-transitive (since it is the underlying chain of the power series field $\mathbb{R}((\mathbb{R}))$).

Now apply Proposition 3.

Thus $\mathbb{R}^{\mathbb{R}+\mathbb{R}} \simeq \mathbb{R}^{\mathbb{R}^{<0}} \vec{\amalg} \mathbb{R}^{\mathbb{R}^{\geq 0}} \simeq \mathbb{R}^\mathbb{R}$ (by AR(9)).

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Part IV.

Algebraic motivation and applications in [K-K-S1] and [K-K-S2].

While trying to define an exponential function on a power series field, we encountered the question

When does Γ embed convexly in Δ^Γ?

Theorem [K-K-S2]

Let Γ and Δ_γ, $\gamma \in \Gamma$, be nonempty chains. For every $\gamma \in \Gamma$, fix a base point 0_γ which is not the last element in Δ_γ. Suppose that Γ has no last element and that Γ' is a cofinal subset of Γ. Then there is no convex embedding

$$\iota : \Gamma' \to \mathop{\mathbf{H}}_{\gamma\in\Gamma} \Delta_\gamma .$$

Corollary 1

Let Γ and Δ be nonempty chains without last element, and fix a base point 0 in Δ. Then there is *no* embedding

$$\iota : \Gamma \to \Delta^\Gamma$$

for which $\iota(\Gamma)$ is convex in Δ^Γ.

Corollary 2

Let G be a nontrivial ordered abelian group and $K = \mathbb{R}((G))$. Then $(K, <)$ admits *no* exponential.

Proof

If K admits an exponential, then $G \simeq \mathbb{R}((G^{<0}))$, as ordered groups. This gives rise to an embedding of $G^{<0}$ in $\mathbb{R}((G^{<0}))$ with convex image. Contradiction.

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