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# *Quasi-Orders: a uniform approach to orders and valuations*

In model-theoretic algebra the classes of **ordered algebraic structures** / **valued structures** play a fundamental role:

- (totally) ordered sets / ultrametric spaces
- (totally) ordered abelian groups / valued abelian groups
- (totally) ordered fields / valued fields

The aim of this talk is to present a uniform approach to the ordered respectively valued cases.

## 1 Quasi-Orders

• A **quasi-order (q.o.)** on a set $S$ is a binary relation $\preceq$ which is reflexive and transitive. An **order** is a q.o which is in addition anti-symmetric.

• Here, we will deal only with **total quasi-order**, i.e. either $a \preceq b$ or $b \preceq a$, for any $a,\ b \in S$.

• The **induced equivalence relation** is defined by $a \asymp b$ if and only if ($a \preceq b$ and $b \preceq a$). We shall write $a \prec b$ if $a \preceq b$ but $b \asymp a$ fails.

• $\preceq$ induces canonically a total order on $S/\asymp$. Conversely if $\asymp$ is an equivalence relation on $S$ such that $S/\asymp$ is a total order, then $\asymp$ induces canonically a q.o. on $S$.

• A subset $E$ of $S$ is **$\preceq$-convex** if for all $a, b, c$ in $S$, if $a \preceq c \preceq b$ and $a,\ b \in E$, then $c \in E$.

## 2 Quasi-Ordered Fields

• A **quasi-ordered field** $(K, \preceq)$ is a field $K$ endowed with a quasi-order $\preceq$ which satisfies the following compatibility conditions, for any $a, b, c \in K$.

**qo1** If $a \asymp 0$, then $a = 0$.

**qo2** If $0 \preceq c$ and $a \preceq b$, then $ac \preceq bc$.

**qo3** If $a \preceq b$ and $b \not\asymp c$ , then $a + c \preceq b + c$.

*Examples:* An ordered field $(K, \leq)$ is a q.o. field. The valuation on a valued field $(K, v)$ induces a quasi-order: $a \preceq_v b$ if and only if $v(b) \leq v(a)$.

• Conversely, Fakhruddin showed that if $\preceq$ is a q.o. on $K$, then $\preceq$ is either an order or there is a (unique up to equivalence of valuations) valuation $v$ on $K$ such that $\preceq = \preceq_v$.

*As an illustration, we re-consider two important problems from classical real algebra and valuation theory:*

*I. Fix an order on a field and then, study all valuations which are compatible with this order (convex valuations, rank of the ordered field)*

*II. Fix a valuation on a field and then, study all orderings which are compatible with it (Baer-Krull theorem, lifting orderings from the residue field)*

Here we shall study Problem I. above but with a "quasi-order" instead of an order...

## 3 Compatible Valuations

Fix a q.o. $\preceq$ on $K$ . Given a valuation $w$ on $K$ , denote the valuation ring by $K_w$, its group of units $K_w^\times$ by $\mathcal{U}$, its unique maximal ideal by $I_w$, the value group by $w(K^\times)$ and residue field $K_w/I_w$ by $Kw$. The valuation $w$ is called

- **convex** with respect to $\preceq$ if $K_w$ is convex.
- **compatible** with $\preceq$ if for all $a, b \in K$ :

$$0 \preceq b \preceq a \quad \Longrightarrow \quad w(a) \leq w(b)\ .$$

- Equivalently, $w$ is compatible with $\preceq$ if and only if for all $a, b \in K$ :

$$0 \preceq b \preceq a \quad \Longrightarrow \quad b \preceq_w a\ .$$

**Remark 3.1** (i) If $\preceq$ is an order, then this is the usual notion of compatibility for orders and valuations.

(ii) If $\preceq = \preceq_v$ is a p.q.o. then $w$ compatible with $\preceq_v$ just means that for all $a, b \in K$ :

$$v(a) \leq v(b) \quad \Longrightarrow \quad w(a) \leq w(b) \, .$$

This in turn just means that $K_v \subseteq K_w$, i.e. that $w$ is a **coarsening** of $v$.

(iii) For $K$ a field endowed with two valuations $v$, $w$, $w$ is coarser than $v$ if and only if $a \preceq_v b$ implies $a \preceq_w b$ , equivalently $\asymp_w$ is coarser than $\asymp_v$. ( If $\sim_1$ and $\sim_2$ are two equivalence relations defined on the same set, then $\sim_1$ is said to be **coarser** than $\sim_2$ if $\sim_2$-equivalence implies $\sim_1$-equivalence).

*The following gives the characterization of valuations compatible with a quasi-order.*

**Theorem 3.2** *Let $(K, \preceq)$ be a q.o. field and $w$ a valuation on $K$. The following assertions are equivalent:*

*1) $w$ is compatible with $\preceq$,*

*2) $w$ is convex,*

*3) $I_w$ is convex,*

*4) $I_w \prec 1$,*

*5) the quasi-order $\preceq$ induces canonically via the residue map $a \mapsto aw$ a quasi-order on the residue field $Kw$.*

• We note that If $\preceq$ is an order then the induced quasi-order in 5) is also an order, if $\preceq$ is a p.q.o then the induced quasi-order in 5) is also a p.q.o.

• Theorem 3.2 is in complete analogy to the characterization of valuations compatible with an order.

• We prove only the p.q.o. case:

Proof: Assume $\preceq=\preceq_v$ is a p.q.o. Compatible valuations are clearly convex, this follows from the definitions. Conversely if $w$ is convex and $0 = v(1) \leq v(a)$ , i.e. $a \preceq 1$ , then $a \in K_w$ by convexity. So $w$ is a coarsening of $v$. This establishes the equivalence of **1) and 2).**

If $w$ is convex, $a \preceq b$ with $b \in I_w$ , then $0 < w(b) \leq w(a)$ by compatibility, so $a \in I_w$. Conversely assume $I_w$ convex, and let $a \preceq b$ with $b \in K_w \setminus I_w$. If $a \notin K_w$ then $a^{-1} \in I_w$. Now $b^{-1} \preceq a^{-1}$, so $b^{-1} \in I_w$ , a contradiction This establishes the equivalence of **2) and 3)**.

If $I_w$ is convex, then $w$ is a coarsening of $v$, so $I_w \subseteq I_v \prec 1$. Conversely, assume $I_w \prec 1$ and let $a \preceq b$ with $b \in K_w$ . If $a \notin K_w$ , then $a^{-1} \in I_w$ . So $a^{-1}b \in I_w$ whence $a^{-1}b \prec 1$. Multiplying by $a$ gives $b \prec a$, a contradiction. This establishes the equivalence of **3) and 4)**.

Now let $w$ be a coarsening of $v$ . Then $v$ induces canonically a valuation $v/w$ on the residue field $Kw$, defined by $v/w(aw) := \infty$ if $aw = 0$ and $v/w(aw) := v(a)$ otherwise. The p.q.o. $\preceq_{v/w}$ is precisely the induced well defined quasi-order in 5), i.e. $aw \preceq_{v/w} bw$ if and only if $a \preceq_v b$ holds. Conversely, let $\preceq_{v/w}$ be a p.q.o. on $Kw$ induced by the residue map. This means that $aw \preceq_{v/w} bw$ if and only if $a \preceq_v b$ holds. Then $w$ is a coarsening of $v$. This establishes the equivalence of **1) and 5)**. □

## 4 The rank of a quasi-ordered field:

I. Let $(K, <)$ be an ordered field.

• The **natural valuation on the ordered field** is the valuation $v$ whose valuation ring $K_v$ is the convex hull of $\mathbb{Q}$ in $K$. It is the finest $<$ - convex valuation of $K$. It is characterized by the fact that its residue field $Kv$ is archimedean, i.e. the only archimedean equivalence classes are those of 0 and 1.

• If $w$ is a coarsening of a convex valuation, then $w$ also is convex. Conversely, a convex subring containing 1 is a valuation ring.

• The set $\mathcal{R}$ of all valuation rings $K_w$ of convex valuations $w \neq v$ (i. e. all strict corsenings of $v$) is totally ordered by inclusion. Its order type is called the **rank of the ordered field** $K$.

• Theorem 3.2 is a characterization of the rank of the ordered field $(K, <)$.

II. Let $(K, \preceq)$ is p.q.o.

• The unique valuation $v$ such that $\preceq = \preceq_v$ is the **natural valuation on the p.q.o. field**. The natural valuation is the finest $\preceq$- convex valuation of $K$.

• A compatible valuation $w$ is a coarsening of $v$. Thus, Theorem 3.2 is a characterization of the **rank of the valued field** $(K, v)$, i. e. the order type of the totally ordered set $\mathcal{R}$ of all strict corsenings of $v$.

• As we recalled in the proof of Theorem 3.2, the natural valuation $v$ induces canonically a valuation $v/w$ on the residue field $Kw$ and $v$ is the **compositum** of $w$ and $v/w$ .

• The p.q.o. $\preceq_{v/w}$ is precisely the induced quasi-order in Theorem 3.2 5). If $w = v$ , then $v/w$ is trivial. Thus $v$ is characterized by the fact that the induced p.q.o on its residue field $Kv$ is **trivial**, i.e. the only equivalence classes of $\asymp$ are those of 0 and 1.