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http://math.usask.ca/~skuhlman/slidespnrms2009.pdf

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An uncountable family of logarithmic functions of distinct growth rates.

Preliminaries.

Let $G \neq 1$ be an ordered abelian group.

• $\mathbb{R}((G))$ will denote the **field of generalized series** with real coefficients, of which support is an anti well ordered and countable subset of G.

• $f = \Sigma_{g \in G} f_g \, g$ with $f_g \in \mathbb{R}$ and

$$\text{supp}\ (f) := \{g \in G \,;\, f_g \text{ nonzero } \}$$

is countable and anti-wellordered.

• Pointwise addition, convolution formula for multiplication of series, anti-lexicographic order, natural valuation.

Denote by $G^{\succ 1}$ the semigroup of elements greater than 1.

• $\mathbb{R}((G^{\succ 1}))$ consists of "purely infinite" series with countable support in $G^{\succ 1}$.

• $\mathbb{R}((G^{\preceq 1}))$ and $\mathbb{R}((G^{\prec 1}))$ denote respectively the valuation ring of bounded elements, and the valuation ideal of infinitesimal elements of $\mathbb{R}((G))$.

• If G is divisible, $\mathbb{R}((G))$ is a (non-archimedean) **real closed field**, i.e. by **Tarski's Tranfer Principle**, $\mathbb{R}((G))$ is elementarily equivalent to the ordered field of real numbers $(\mathbb{R}, <)$.

• A. Wilkie's o-minimality of $(\mathbb{R}, \log)$.

• *How to construct nonarchimedean logarithmic fields using fields of generalized series?*

• Use Taylor expansion of the logarithm to define the logarithm of a generalized series?

Summable families of series: Given a family

$$\{s_i\,; i \in I\} \subset \mathbb{R}((G))$$

make sense of $\sum_{\imath \in I} s_i$ as an element of $\mathbb{R}((G))$.

Defining the logarithm.

• **B.H.Neumann:** For $\epsilon \in \mathbb{R}((G^{\prec 1}))$,

$$\sum_{i=1}^{+\infty} (-1)^{(i-1)} \frac{\epsilon^i}{i}$$

makes sense.

A **logarithmic section** is an embedding of ordered groups

$$l : (G\ , \cdot, \prec) \to (\mathbb{R}((G^{\succ 1})), +)\ .$$

• Given $f \in \mathbb{R}((G))$, $f > 0$ and $g :=\ \max\ \mathrm{supp}\ f$, write

$$f = g \cdot c \cdot (1 + \epsilon)$$

with $c \in \mathbb{R}$, $c > 0$, $\epsilon \in \mathbb{R}((G^{\prec 1}))$.

• We extend l as follows:

$$l(f) = l(g \cdot c \cdot (1 + \epsilon)) = l(g) + \log c + \sum_{i=1}^{+\infty} (-1)^{(i-1)} \frac{\epsilon^i}{i}$$

• $l : (\mathbb{R}((G))^{>0}, \cdot) \to (\mathbb{R}((G)), +)$ is an order preserving embedding of groups, extending the logarithmic section l (the **logarithm** associated to the logarithmic section l).

Logarithmic sections from Hahn groups

Let us now consider a totally ordered set Γ, we now explain how this data determines a logarithmic section:

• Consider the multiplicative group G which consists of finite products of germs f^r, $f \in \Gamma$, $r \in \mathbb{R}$.

• Consider $l : G \to \mathbb{R}((G))$ defined by

$$l(\prod_{i=1}^{s} f_i^{r_i}) := \sum_{i=1}^{s} r_i f_i \ ,$$

defines indeed a logarithmic section on $\mathbb{R}((G))$.

• But this logarithmic section violates the **growth axiom**. We need more.

• We assume that Γ admits an order preserving automorphism which is a **leftward shift:**

$$\sigma(f) \prec f \text{ for all } f \in \Gamma \ .$$

• The automorphism σ induces the logarithmic section:

$$l_\sigma(\prod_{i=1}^{s} f_i^{r_i}) := \sum_{i=1}^{s} r_i \sigma(f_i) \ .$$

• We extend l_σ to a logarithm defined on $\mathbb{R}((G))$ as before.

Rank and logarithmic rank

We see that pairwise distinct left shifts on Γ will induce pairwise distinct logarithms. We do more: we construct logarithms of pairwise distinct growth rates.

The **rank** of (Γ, σ) is the order type of the quotient $\Gamma / \sim_\sigma$, where $a \sim_\sigma a'$ if and only if there exists $n \in \mathbb{N}$ such that $\sigma^{(n)}(a) \geq a'$ and $\sigma^{(n)}(a') \geq a$.

Similarly the **logarithmic rank** of $(K^{>0}, l)$ is defined via the equivalence relation: a, $a' \in K^{>0}$ are *log-equivalent* if $a \sim_l a'$, that is, if and only if there exists

$$n \in \mathbb{N} \text{ such that } l^{(n)}(a) \leq a' \text{ and } l^{(n)}(a') \leq a \,.$$

Proposition 0.1 *The logarithmic rank of* $(\mathbb{R}((G)), l_\sigma)$ *is equal to the rank of* (Γ, σ).

An asymptotic scale indexed by $\aleph_1 \times \mathbb{Z}^2$.

We construct a totally ordered set of germs at infinity of real valued functions of a real variable, which admits $2^{\aleph_1}$ *left shifts.*

• For $(p, q) \in \mathbb{Z}^2$, we denote by $g_{p,q}$ the germ at $+\infty$ of the infinitely large *transmonomial*

$$x \mapsto \exp\left(x^q \exp\left(x^p\right)\right).$$

If we endow $\mathbb{Z}^2$ with the lexicographic order, then $(p, q) < (p', q')$ implies $g_{p,q} \prec g_{p',q'}$.

• Now let $\{h_\alpha\,; \alpha \in \aleph_1\}$ be a sequence of germs at $+\infty$ of infinitely large transmonomials h_α, in such a way that $\alpha < \beta$ implies $h_\alpha \prec h_\beta$.

• One can describe for example the first ϵ_0 terms of such a sequence. Set $h_0(x) := x$. We define h_α by transfinite induction for $\alpha < \epsilon_0$. If the Cantor normal form of α is $\omega^{\beta_r} d_r + \cdots + \omega^{\beta_1} d_1 + d_0$, with $\beta_1 < \cdots < \beta_r < \alpha$ and $d_{0,} \ldots, d_r \in \mathbb{N}$, set

$$h_\alpha\left(x\right) := \exp\left(d_r h_{\beta_r}\left(x\right) + \cdots + d_1 h_{\beta_1}\left(x\right)\right) \exp(x)^{d_0} \,.$$

We can set $h_{\epsilon_0} := t(x)$ where $t(x)$ is a germ of transexponential growth.

• Finally: for all $(\alpha, p, q) \in \aleph_1 \times \mathbb{Z}^2$, we denote $f_{\alpha,p,q}$ the germ at $+\infty$ of the transmonomial $\exp_3 (h_\alpha (x))\, g_{p,q} (x)$.

• These germs are defined in such a way that if $(\alpha, p, q) < (\alpha', p', q')$ for the lexicographic order, then $f_{\alpha,p,q} \prec f_{\alpha',p',q'}$. This set of germs Γ is thus totally ordered.

We construct $2^{\aleph_1}$ left-shifts of pairwise distinct ranks on Γ. To this end, we consider the two automorphisms defined on $\Gamma_1 = \left\{ g_{p,q}\,,\ (p,q) \in \mathbb{Z}^2 \right\}$ by :

$$\begin{aligned} \sigma\left(g_{p,q}\right) &= g_{p-1,q} \\ \rho\left(g_{p,q}\right) &= g_{p,q-1} \end{aligned}$$

It follows easily from the definition of $g_{p,q}$ that the rank of (Γ_1, σ) is 1 and the rank of (Γ_1, ρ) is $\mathbb{Z}$. We define now, for every $S \subset \aleph_1$, the decreasing automorphism τ_S on Γ by :

$$\tau_S\left(f_{\alpha,p,q}\right) = \begin{cases} f_{\alpha,p-1,q} = \exp_3\left(h_\alpha\right)\sigma\left(g_{p,q}\right) & \text{si } \alpha \in \mathrm{S} \\ f_{\alpha,p,q-1} = \exp_3\left(h_\alpha\right)\rho\left(g_{p,q}\right) & \text{si } \alpha \notin \mathrm{S} \end{cases}$$

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