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The slides of this talk are available at:
http://math.usask.ca/~skuhlman/slideshelton2010.pdf

The General Moment Problem.

The Multidimensional Moment Problem.

• Let $V := \mathbb{R}[x] := \mathbb{R}[x_1, \cdots, x_n]$ be the real vector space of polynomials in n variables and real coefficients.

In analogy to the classical Riesz Representation Theorem, Haviland considered the problem of representing linear functionals on V by measures. The question of when, given a closed subset $K \subseteq \mathbb{R}^n$, a linear map $\ell : \mathbb{R}[x] \to \mathbb{R}$ corresponds to a finite positive Borel measure μ on K is known as the Multidimensional Moment Problem.

• Define the *cone of nonnegative polynomials* on K by

$$\mathrm{Psd}(K) = \{f \in \mathbb{R}[x] \ : \ \forall x \in K \ f(x) \geq 0\}.$$

In 1935, he proved the following :

Theorem (Haviland)

Let $K \subset \mathbb{R}^n$ closed, and $\ell : V \to \mathbb{R}$ a nonzero linear functional. The following are equivalent:

(i) $\ell(f) \geq 0$ for all $f \in \mathrm{Psd}(K)$

(ii) $\exists$ a positive Borel measure μ on K such that

$$\ell(f) = \int_K f d\mu \, , \forall \, f \in V$$

The main challenge in applying Haviland's Theorem is verifying its condition (i). Schmüdgen analysed this problem for a special class of closed subsets:

• $K \subseteq \mathbb{R}^n$ is a *basic closed semialgebraic set* if there exist a finite set of polynomials $S = \{g_1, \ldots, g_s\}$ such that

$$K = K_S := \{x \in \mathbb{R}^n \; : \; g_i(x) \geq 0, \; i = 1, \ldots, s\}.$$

• Consider the cone $T_S \subseteq \mathrm{Psd}(K)$:

$$T_S := \{ \sum_{e \in \{0,1\}^s} \sigma_e \underline{g}^e \; : \; \sigma_e \text{ is a sos for all } e \in \{0,1\}^s \},$$

where $e = (e_1, \cdots, e_s) \in \{0,1\}^s$, and

$\underline{g}^e := \; g_1^{e_1} \ldots g_s^{e_s}$.

In 1991 Schmüdgen improved condition (i) of Haviland's Theorem and proved the following:

Theorem (Schmüdgen)

Assume that $K = K_S$ is a *compact* basic closed semi-algebraic set, and $\ell : V \to \mathbb{R}$ a nonzero linear functional. The following are equivalent:

(i) $\ell(h^2 \underline{g}^e) \geq 0 \quad \forall h \in \mathbb{R}[x]$ and $e \in \{0,1\}^s$

(ii) $\exists$ a positive Borel measure μ on K such that

$$\ell(f) = \int_K f d\mu \,, \forall\, f \in V$$

Thus condition (i) reduces to verifying 2^s schemes, or equivalently the psd-ness of 2^s infinite Hankel matrices.

Putinar reduces further to just $s + 2$.

Various results improving this number by considering special properties of the defining polynomials or of the semi-algebraic set, such as exploiting symmetry, sparsity, convexity, etc....

Here, we take a natural step in a different direction by exploiting special properties of the linear functionals under consideration.

• Given ℓ we consider the sequence of evaluations on the monomial basis:

$$s(\alpha) := \ell(x^{\alpha})\,;\alpha \in \mathbb{N}^n$$

We shall read off this sequence properties of ℓ such as continuity.

Closures of Cones in Locally Convex Topologies.

• Fix τ a locally convex topological vector space topology on V. Denote V_τ the corresponding topological space.

Let $C \subseteq V$ be a cone (i.e. closed under addition and scalar multiplication by positive reals). Define

• The dual of C:

$$C^\vee := \{\ell \mid \ell : V_\tau \to \mathbb{R}\,; \text{cts linear functional}; \ell(C) \geq 0\}$$

• The double dual of C:

$$C^{\vee\vee} := \{f \in V \mid \ell(f) \geq 0 \ \forall \ \ell \in C^\vee\}$$

• Since $C \subset V$ is a (convex) cone, we have

$$C^{\vee\vee} = \overline{C}$$

in V_τ (Hahn–Banach).

We use Haviland's theorem and the properties of duality and closures to deduce the following:

Corollary 1 Let τ be a locally convex topology on V, $C \subseteq V$ a cone, $K \subseteq \mathbb{R}^n$ a closed subset. The following are equivalent:

(1) $\overline{C} = \text{Psd}(K)$ in V_τ

(2) for a continuous linear functional ℓ ; $\ell(C) \geq 0$ if and only if $\exists\ \mu$ on K such that:

$$\ell(f) = \int_K f d\mu \, , \forall\, f \in V$$

Example: For $\tau = \varphi :=$ the finest locally convex topology, all linear functionals are continuous. Schmüdgen's result can be reformulated as:

Let $K = K_S$ be a compact basic closed semi-algebraic set. Then

$$\overline{T_S} = \text{Psd}(K) \text{ in } V_\varphi \, .$$

Are there other interesting examples?

The Moment Problem for Continuous Positive Semidefinite Linear Functionals.

In the following, we shall study situations where the 2^s conditions (i) in Schmüdgen can be replaced by the single condition

$$\ell(h^2) \geq 0 \text{ for all } h \in \mathbb{R}[x] \,.$$

Call a linear functional ℓ *positive semi definite* if this condition holds.

Below, for $1 \leq p \leq \infty$:

$V_p := V$ endowed with the ℓ_p–norm topology (on the coefficients of polynomials).

Theorem (Berg et al.):

$$\overline{\sum V^2} = \text{ Pos } [-1,1]^n \text{ in } V_1 \,.$$

Corollary Let ℓ be a continuous linear functional on V_1 (i.e. the sequence $(\ell(x^\alpha))_{\alpha\in\mathbb{N}^n}$ is bounded).

Assume that ℓ is positive semi-definite. Then

$$\exists \mu \text{ on } [-1,1]^n \text{ such that } \ell(f) = \int f d\mu \;\forall\; f \in V \,.$$

Remark Compare to Schmüdgen: We can describe the compact basic closed semi-algebraic unit hypercube by $2n$ linear inequalities. for an arbitrary linear functional, we would a priori check 2^{2n} Hankel matrices.

Weighted ℓ_p Topologies.

Let $r = (r_1, \ldots, r_n)$ be a n-tuple of positive real numbers.

- For $1 \leq p < \infty$,

$$\ell_{p,r}(\mathbb{N}^n) := \{s \in \mathbb{R}^{\mathbb{N}^n} : \sum_{\alpha \in \mathbb{N}^n} |s(\alpha)|^p r_1^{\alpha_1} \ldots r_n^{\alpha_n} < \infty\}$$

is a Banach space with respect to the norm

$$\|s\|_{p,r} = (\sum_{\alpha \in \mathbb{N}^n} |s(\alpha)|^p r_1^{\alpha_1} \ldots r_n^{\alpha_n})^{\frac{1}{p}}.$$

- For $p = \infty$

$$\ell_{\infty,r}(\mathbb{N}^n) := \{s \in \mathbb{R}^{\mathbb{N}^n} : \sup_{\alpha \in \mathbb{N}^n} |s(\alpha)| r_1^{\alpha_1} \ldots r_n^{\alpha_n} < \infty\}$$

is a Banach space with respect to the norm

$$\|s\|_{\infty,r} = \sup_{\alpha \in \mathbb{N}^n} |s(\alpha)| r_1^{\alpha_1} \ldots r_n^{\alpha_n}.$$

Let us describe the continuous linear functionals on $\ell_{p,r}(\mathbb{N}^n)$. Below, we let q be the conjugate of p.

Proposition. Let $1 \leq p < \infty$.
If $p > 1$, then $\ell_{p,r}(\mathbb{N}^n)^* = \ell_{q,r^{-\frac{q}{p}}}(\mathbb{N}^n)$.
If $p = 1$, then $\ell_{1,r}(\mathbb{N}^n)^* = \ell_{\infty,r^{-1}}(\mathbb{N}^n)$.

Here $r^{-\frac{q}{p}} := (r_1^{-\frac{q}{p}}, \cdots, r_n^{-\frac{q}{p}})$, similarly for r^{-1}.

Now let $f \in V$. Assume that

$$f \geq 0 \text{ on } \prod_{i=1}^{n} [-r_i, r_i] \, .$$

Then the polynomial $\tilde{f}(\underline{X}) = f(r_1 X_1, \cdots, r_n X_n)$ is a non-negative polynomial on $[-1, 1]^n$.

Combining this observation with Berg's result we get:

Fix $r = (r_1, \cdots, r_n)$ with $r_i > 0$ for $i = 1, \cdots, n$.

Theorem 1 Let $p = 1$. Then

$$\overline{\sum V^2} \;=\; \text{Psd}\ (\prod_{i=1}^{n} [-r_i, r_i]) \text{ in } V_{1,r}\ .$$

We further generalize:

Theorem 2 Let $1 < p < \infty$. Then

$$\overline{\sum V^2} = \text{Psd}\ (\prod_{i=1}^{n} [-r_i^{\frac{q}{p}}, r_i^{\frac{q}{p}}]) \text{ in } V_{p,r}\ .$$

Here, for $1 \leq p \leq \infty$:

$V_{p,r} := V$ endowed with the $\ell_{p,r}$–norm topology (on the coefficients of polynomials).

Corollary 1 Let $\ell : \mathbb{R}[x] \to \mathbb{R}$ be a linear functional such that the sequence $s(\alpha) = \ell(x^\alpha)$ satisfies

$$\sup_{\alpha \in \mathbb{N}^n} |s(\alpha)| r_1^{-\alpha_1} \cdots r_n^{-\alpha_n} < \infty \, .$$

Then ℓ is positive semidefinite if and only if there exists a positive Borel measure μ on $K = \Pi_{i=1}^n [-r_i, r_i]$ such that

$$\ell(f) = \int_K f \; d\mu \quad \forall f \in \mathbb{R}[x] \, .$$

Corollary 2 Let $1 < p < \infty$.
Let $\ell : \mathbb{R}[x] \to \mathbb{R}$ be a linear functional such that the sequence $s(\alpha) = \ell(x^\alpha)$ satisfies

$$\sum_{\alpha \in \mathbb{N}^n} |s(\alpha)|^q r_1^{-\frac{q}{p}\alpha_1} \cdots r_n^{-\frac{q}{p}\alpha_n} < \infty . \, .$$

Then ℓ is positive semidefinite if and only if there exists a positive Borel measure μ on $K = \Pi_{i=1}^n [-r_i^{-\frac{q}{p}}, r_i^{-\frac{q}{p}}]$ such that

$$\ell(f) = \int_K f \; d\mu \quad \forall f \in \mathbb{R}[x] \, .$$

In the particular case where $r_1 = \cdots = r_n$, we deduce the result of Berg and Maserick on "exponentially bounded" positive semidefinite moment sequences. In fact, in this case, the condition in Corollary 1 implies the existence of a positive real number R such that

$$|s(\alpha)| \leq R r_1^{\alpha_1 + \cdots + \alpha_n}.$$

Hence implies that ℓ can be represented as an integral with respect to a measure on $[-r_1, r_1]^n$.

Furture Work: Let K be a (compact? convex? polyhedral?) basic closed semi algebraic subset of $\mathbb{R}^n$, and ℓ a positive semidefinite linear functional on V. Find a (checkable!) necessary and sufficient condition on the sequence $s(\alpha)$ so that ℓ is represented by a positive Borel measure on K.

Procedure: Given the defining inequalities of K, try to construct a locally convex toplogy τ such that

$$\overline{\sum V^2} = \text{ Psd } (K) \text{ in } V_\tau \, .$$

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