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Fields of generalized power series.

I. Lexicographic powers of ordered sets.

Let Δ and Φ be (*linearly i.e. totally*) ordered sets. Fix an distinguished element $0 \in \Delta$. The lexicographic power Δ^Φ is the following set:

$$\begin{aligned} \Delta^\Phi &:= \{s : \Phi \to \Delta \,;\, \text{support}\, s \text{ is well-ordered}\} \\ &= \{s \in \prod_{\phi \in \Phi} \Delta \,;\, \text{support}\, s \text{ is well-ordered}\}\,, \end{aligned}$$

ordered lexicographically from the left,

that is “order by first differences”.

Here support $s := \{\phi \in \Phi \,;\, s_\phi \neq 0\}$.

F. Hausdorff and others studied their order types, generalizing Cantor's ordinal arithmetic and construction of saturated and universal models (the so-called $\eta_\alpha -$*sets) for the theory of dense linear ordering without endpoints.*

• If α are β ordinals, then the lexicographic power α^{β^*} has order type the ordinal α^{β}.

• Hausdorff's $\eta_\alpha-$set is constructed with the lexicographic power $2^{\aleph_\alpha}$.

• Other Examples: $\mathbb{Z}^{\mathbb{N}}$ has the order type of the set of irrationals, $\mathbb{N}^{\mathbb{N}}$ that of the set non-negative real $\mathbb{R}^+$, $2^{\mathbb{N}}$ that of the Cantor set.

Many fascinating problems (studied with W.C. Holland and S. McCleary) such as: dependence on the choice of the distinguished element 0*, isomorphism of powers with same base but different exponents or vice-versa, etc....*

II. Lexicographically ordered abelian goups.

If Δ is an ordered abelian group, e.g. $\Delta = \mathbb{R}$ for simplicity, we can endow the lexicographic power $\mathbb{R}^\Phi$, which we then denote by $\mathbf{H}_\Phi\mathbb{R}$, with an ordered abelian group structure.

Indeed, using now for $s \in \mathbb{R}^\Phi$ the notation

$$s = \sum_\phi s_\phi \mathbf{1}_\phi$$

define pointwise addition:

$$s+r = \sum_\phi s_\phi \mathbf{1}_\phi + \sum_\phi r_\phi \mathbf{1}_\phi := \sum_\phi (s_\phi + r_\phi) \mathbf{1}_\phi \, .$$

Obviously, the support of $s+r$ is still well-ordered, so $s+r$ is well-defined.

H. Hahn and others introduced and studied these so-called Hahn-groups. They are used for constructing saturated and universal models for the theory of divisible ordered abelian groups:

Theorem [Hahn Embedding's Theorem]:
Every ordered abelian group Γ is isomorphic to a subgroup of a Hahn group $\mathbf{H}_\Phi\mathbb{R}$ for a suitable Φ.

More precisely, Φ is uniquely determined by Γ, it is the so-called archimedean "rank" of Γ. So Hahn's theorem generalizes O.L. Hölder's Theorem to the non-archimedean case.

Theorem [N.L. Alling - S.K.]:
Let Φ be an $\eta_\alpha-$set, then the Hahn group $\mathbf{H}_\Phi\mathbb{R}$ is an $\aleph_\alpha-$saturated divisible ordered abelian group.

Let us continue enriching our lexicographic powers...

III. Lexicographically ordered fields.

If Δ is an ordered field, e.g. $\Delta = \mathbb{R}$ for simplicity, *and* Φ ordered abelian group call it Γ, we can endow the lexicographic power $\mathbb{R}^\Gamma$, which we then denote by $\mathbb{R}((\Gamma))$, with a field structure.

Indeed, using now for $s \in \mathbb{R}^\Gamma$ the notation

$$s = \sum_\gamma s_\gamma t^\gamma$$

define multiplication via convolution:

$$s.r = \sum_\gamma \Big(\sum_{\gamma' + \gamma'' = \gamma} (s_{\gamma'} r_{\gamma''}) \ \Big) t^\gamma \ .$$

Is $s.r$ *well-defined?* That is, is it true that (i) for every $\gamma \in \Gamma$,

$$\sum_{\gamma' + \gamma'' = \gamma} (s_{\gamma'} r_{\gamma''})$$

is a finite sum? and (ii) $s.r$ has well-ordered support? The answer is yes. Why?

IV. Summability

We need the following key notion. Let I be an infinite index set, $F := \{s_i \in \mathbb{R}((\Gamma))\ ;\ i \in I\}$ a family of series, set

$$\mathrm{Support} F := \bigcup_{i \in I} \mathrm{support}\ s_i\ .$$

F is said to be **summable** if:

(i) For any $\gamma \in \mathrm{Support} F$, the set

$$S_\gamma := \{i \in I \mid \gamma \in \mathrm{support}\ s_i\} \subseteq I$$

is finite.

(ii) Support$F$ is well-ordered.

Write $s_i = \sum_\gamma (s_i)_\gamma t^\gamma$ for each $s_i \in \mathcal{F}$. If $\mathcal{F}$ is summable. Then

$$\sum_{i \in I} s_i := \sum_{\gamma \in \text{Support } \mathcal{F}} (\sum_{i \in S_\gamma} (s_i)_\gamma) t^\gamma$$

is a well-defined element of $\mathbb{R}((\Gamma))$ that we call the **sum** of $\mathcal{F}$.

Returning to multiplication: $s.r$ is well-defined because one can verify that the family

$$\{ t^{\gamma'}.r \ ; \ \gamma' \in \text{support } s \}$$

is summable.

W.Krull, I. Kaplansky and others studied the field $\mathbb{R}((\Gamma))$ *while developing valuation theory, again we have universality and saturation:*

V. Non-archimedean Real Closed Fields

In what follows, we assume always Γ is non-trivial, i.e. $\Gamma \neq 0$.
If Γ is divisible, then $\mathbb{R}((\Gamma))$ is a non-standard model of $\mathrm{Th}(\mathbb{R}, 0, 1, +, \times, <)$.

Theorem [Kaplansky Embedding's Theorem]:
Every real closed field R is isomorphic to a subfield of a field of generalized series $\mathbb{R}((\Gamma))$ for a suitable Γ.

More precisely, Γ is uniquely determined by R, it is the so-called value group of R.

Theorem [N.L. Alling - S.K.]:
Let Γ be an $\aleph_\alpha-$saturated divisible ordered abelian group. Then the field of generalized series $\mathbb{R}((\Gamma))$ is an $\aleph_\alpha-$saturated real closed field.

We have studied dense linear orderings without endpoints, divisible ordered abelian groups, real closed fields, all are so-called o-minimal structures. Let us continue enriching our fields of power series with further o-minimal structure....

VI. Exponentiation

The subring $\mathbb{R}((\Gamma^{\geq 0}))$ of $\mathbb{R}((\Gamma))$ (consisting of series with support contained in the non-negative cone $\Gamma^{\geq 0}$ of the Γ) is a valuation ring, with a unique maximal ideal $\mathbb{R}((\Gamma^{+}))$ consisting of infinitesimal series, i.e. series with strictly positive support.

Theorem [Neumann's Lemma]

Let $\epsilon \in \mathbb{R}((\Gamma^{+}))$ and $c_i \in \mathbb{R}, i \in \mathbb{N}$. Then $\{c_i \epsilon^i \ ; \ i \in \mathbb{N}\}$ is summable. In particular one can define $f(\epsilon)$ for any real analytic function.

We define an exponential function on $\mathbb{R}((\Gamma^{+}))$:

$$\exp(\epsilon) := \sum \frac{\epsilon^i}{i!}$$

and its inverse map on the multiplicative group of 1- units $1 + \mathbb{R}((\Gamma^{+}))$:

$$\log(1+\epsilon) := \sum (-1)^{i-1} \ \frac{\epsilon^i}{i}$$

.

How to define a total surjective exponential function on $\mathbb{R}((\Gamma))$*, i.e. an ordering preserving isomorphism from the ordered additive group* $(\mathbb{R}((\Gamma)), +)$ *onto the ordered multiplicative group* $(\mathbb{R}((\Gamma))^{>0}, \times)$*?*

VII. Lexicographic Decomposition

Theorem [S.K.]

We have the following direct sum (respectively, multiplicative direct sum) decompositions:

$$\begin{array}{rcccccc} \mathbb{R}((\Gamma)) & = & \mathbb{R}((\Gamma^-)) & \oplus & \mathbb{R} & \oplus & \mathbb{R}((\Gamma^+)), \\ \mathbb{R}((\Gamma))^{>0} & = & t^\Gamma & \times & \mathbb{R}^+ & \times & (\,1 + \mathbb{R}((\Gamma^+))\,). \end{array}$$

Indeed given $s \in \mathbb{R}((\Gamma))$ write

- $s = s_{<0} + s_0 + s_{>0}$ and
- for $s > 0$ and $\gamma :=$ min support s, write

$$s = t^\gamma \cdot c \cdot (1 + \epsilon)$$

with $c \in \mathbb{R}$, $c > 0$, $\epsilon \in \mathbb{R}((\Gamma^+))$.

VIII. Consequences of the Lexicographic Decomposition

1. Left Exponentiation? We see that it is necessary and sufficient to construct a **left logarithm**, that is, an ordering preserving isomorphism from the ordered multiplicative group t^Γ onto the ordered additive group $\mathbb{R}((\Gamma^-))$.

Theorem [Kuhlmann-Kuhlmann-Shelah]

There exists a canonical ordering preserving *embedding* from the ordered multiplicative group t^Γ into the ordered additive group $\mathbb{R}((\Gamma^-))$, i.e. a *non-surjective* left logarithm. This embedding cannot be surjective, unless Γ is a proper class. In other words, a field of generalized power series does not admit left-exponentiation.

2. Conway's "Field" of surreal numbers

The "field" **No** *of surreal numbers was invented by J. Conway, studied by H. Gonshor, D. Knuth, M. Kruskal, N.L. Alling, P. Ehrlich and others. It admits left-exponentiation, however it is not a "field" since it is a proper class!*

3. Construction of non-archimedean models of real exponentiation

Consider $T_{\exp} :=$ Th $(\mathbb{R}, 0, 1, +, \times, \exp, <)$.

A. Tarski asked whether $T_{\exp}$ is decidable. A. Wilkie proved that is is model-complete and o-minimal.

We constructed non-standard models of $T_{\exp}$, the **exponential-logarithmic series fields**, as increasing countable union of fields of generalized power series. Indeed since the left-logarithm is not surjective, there are "missing exponentials" in $K_0 := \mathbb{R}((\Gamma_0))$. We enlarge Γ_0 to a Γ_1 so that $K_1 := \mathbb{R}((\Gamma_1))$ contains the "missing exponentials" of K_0. Iterating this procedure, that is, constructing the **exponential closure** of K_0 results in a field $\cup_{\imath \in \mathbb{N}} K_i$ which is now closed under exponentiation.

4. On the decidability of $T_{\exp}$.

A. Macintyre and A. Wilkie showed that $T_{\exp}$ *is decidable if the real* **Schanuel conjecture** *has a positive solution.*

S. Schanuel conectured that if $y_1, \cdots, y_n \in \mathbb{R}$ *are linearly independent over* $\mathbb{Q}$*, then the transcendence degree over* $\mathbb{Q}$ *of the field*

$$\mathbb{Q}(y_1, \cdots, y_n\,;\, \exp(y_1), \cdots, \exp(y_n))$$

is at least n*.*

J. Ax proved Schanuel's conjecture for formal Laurent series without constant term.

Theorem[J. Ax]

Let $y_i \in \mathbb{R}[[t]]$ such that $y_i - y_i(0)$ are $\mathbb{Q}$-linearly independent, $i = 1, \cdots, n$. Then

$$\mathrm{td}_{\mathbb{R}}\mathbb{R}(y_1, \cdots, y_n, \exp(y_1), \cdots, \exp(y_n)) \geq n + 1$$

.

With M. Matusinski and A. Shkop we show that this result holds for exponential-logarithmic series.

5. Integer Parts and Models of Arithmetic.

An **integer part** for an ordered field R is a discretely ordered subring Z such that for each $r \in R$, there exists some $z \in Z$ with $z \leq r < z + 1$.

Shepherdson shows that the class of integer parts of real closed fields coincides with the class of models of **open induction**. *He constructs an integer part of the field of Puiseux series, in which primes are not cofinal. Many open questions about integer parts of real closed fields, and their primes and irreducibles arise naturally.*

Theorem [J.-P. Ressayre and M.-H. Mourgues]
$Z := \mathbb{R}((\Gamma^-)) \oplus \mathbb{Z}$ is an integer part of the real closed field $\mathbb{R}((\Gamma))$.

Proof: Clearly, Z is a discrete subring. Let $s \in \mathbb{R}((\Gamma))$. Let $\lfloor s_0 \rfloor \in \mathbb{Z}$ be the integer part of $s_0 \in \mathbb{R}$. Define

$$z_s = \begin{cases} s_{<0} + s_0 - 1 & \text{if } s_0 \in \mathbb{Z} \text{ and } s_{>0} < 0, \\ s_{<0} + \lfloor s_0 \rfloor & \text{otherwise.} \end{cases}$$

Clearly, $z_s \leq s < z_s + 1$.

• M. Kotchetov we studies primes and irreducibles in integer parts of real closed fields, we showed that this integer part has a cofinal set of primes.

• With M. Carl, P. D'Aquino, L. Gregory, we consider other fragments of Peano Arithmetic:

Does $\mathbb{R}((\Gamma))$ *admit an integer part which is a model of normal open induction, of full Peano Arithmetic*?

We have discussed above Kaplansky's embedding theorem: Fields of generalized series are universal domains for ordered fields. In particular, real closed fields of generalized series provide suitable domains for the study of real algebra. The material presented in the next slides is motivated by the following query: are fields of generalized series suitable domains for the study of real differential algebra? We therefore investigate how to endow a field of generalized series with a "series-derivation".

IX. Defining Derivations.

1. Hahn groups written multiplicatively

Let $(\Phi, \preceq)$ be a totally ordered set, that we call the set of **fundamental monomials**.

Consider the set Γ of formal products $\gamma \in \Gamma$ of the form

$$\gamma = \prod_{\phi \in \Phi} \phi^{\gamma_\phi}$$

where $\gamma_\phi \in \mathbb{R}$, and the support of γ

$$\text{support } \gamma := \{\phi \in \Phi \mid \gamma_\phi \neq 0\}$$

is an anti-well-ordered subset of Φ.

Multiplication of formal products is defined pointwise: for $\alpha, \beta \in \Gamma$

$$\alpha\beta = \prod_{\phi \in \Phi} \phi^{\,\alpha_\phi + \beta_\phi}$$

Γ is an abelian group with identity 1 (the product with empty support).

We endow Γ with the anti lexicographic ordering $\preceq$ which extends $\preceq$ of Φ:

$\gamma \succ 1$ if and only if $\gamma_\phi > 0$, for $\phi := \max(\text{support}\, \gamma)$.

The **leading fundamental monomial** of $1 \neq \gamma \in \Gamma$ is $\mathrm{LF}(\gamma) := \max(\text{support}\ \gamma)$.

Γ is a totally ordered abelian group, the **Hahn group of generalised monic monomials**.

2. Differentiating term by term?.

We want to differentiate

$$a = \sum_{\alpha \in \Gamma} a_\alpha \alpha$$

term by term.

There are two problems:

(i) we first have to know how to differentiate a monomial $\alpha \in \Gamma$,

(ii) then we have to make sense of

$$a' = \sum_{\alpha \in \Gamma} a_\alpha \alpha'$$

a possibly infinite sum of field elements.

In other words, we have summability issues...

3. Series derivations.

Let

$$\begin{array}{rcl} d_\Phi : \Phi & \to & K\backslash\{0\} \\ \phi & \mapsto & \phi' \end{array}$$

be a map.

We say d_Φ **extends to a series derivation on** Γ if the following property holds:

(SD1) For any anti-well-ordered subset $E \subset \Phi$,

the family $\left(\dfrac{\phi'}{\phi}\right)_{\phi \in E}$ is summable.

Then the **series derivation** d_Γ on Γ (extending d_Φ) is defined to be the map

$$d_\Gamma : \ \Gamma \to K$$

obtained through the following axioms:

•[(D0)] $1' = 0$

• [(D1) Strong Leibniz rule:]

$$\text{If } \alpha = \prod_{\phi \in \text{support } \alpha} \phi^{\alpha_\phi} \ \text{ then } (\alpha)' = \alpha \sum_{\phi \in \text{support}\alpha} \alpha_\phi \frac{\phi'}{\phi}.$$

We say that a series derivation d_Γ on Γ **extends to a series derivation on** K if the following property holds:

(SD2) For any anti-well-ordered subset $E \subset \Gamma$, the family $(\alpha')_{\alpha \in E}$ is summable.

Then the **series derivation** d on K (extending d_Γ) is defined to be the map

$$d \; : \; K \to K$$

obtained through the following axiom:

(D2) Strong linearity:

$$\text{If } a = \sum_{\alpha \in \text{Support } a} a_\alpha \alpha, \text{ then } a' = \sum_{\alpha \in \text{Support } a} a_\alpha \alpha'.$$

We now study necessary and sufficient condition on the map d_Φ so that properties (SD1) and (SD2) hold.

4. Sequential Characterization Summability.

We use the following two key observations:

(i) $\mathcal{F}$ is summable if and only if every countably infinite subfamily is summable.

(ii) (Infinite Ramsey.) Let Γ be a totally ordered set. Every sequence $(\alpha_n)_{n\in\mathbb{N}}$ in Γ has an infinite subsequence which is either constant, or strictly increasing, or strictly decreasing.

We isolate the following two crucial "bad" hypotheses:

(H1) There exists a strictly decreasing sequence $(\phi_n)_{n\in\mathbb{N}}$ in Φ and an increasing sequence $(\tau^{(n)})_{n\in\mathbb{N}}$ in Γ such that $\tau^{(n)} \in \text{Support}\ \dfrac{\phi_n'}{\phi_n}$ for all $n \in \mathbb{N}$.

(H2) There exist strictly increasing sequences $(\phi_n)_{n\in\mathbb{N}}$ in Φ and $(\tau^{(n)})_{n\in\mathbb{N}}$ in Γ such that $\tau^{(n)} \in \text{Support}\ \dfrac{\phi_n'}{\phi_n}$ and $\text{LF}\left(\dfrac{\tau^{(n+1)}}{\tau^{(n)}}\right) \succeq \phi_{n+1}$, for all $n \in \mathbb{N}$,

Theorem A: A map $d_\Phi : \Phi \rightarrow K\backslash\{0\}$ extends to a series derivation on K if and only $(H1)$ and $(H2)$ fail.

The End