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Differential valued fields of κ-bounded exponential logarithmic series.

Salma Kuhlmann, Universität Konstanz, Germany.

• **(1902-1912):** F. Hausdorff on foundations of *set theory*, and independently H. Hahn, develop universal constructions of totally ordered abelian groups and fields.

• **(1910):** G. H. Hardy monograph on *asymptotic scales for differentiable* real valued functions.

• **(1927)** E. Artin and O. Schreier develop the algebraic theory of *real closed fields*.

• **(1930)** A. Tarski establishes *quantifier elimination for and decidabilty* of T:= $\mathrm{Th}(\mathbb{R}, +, \cdot, <)$.

• **(1932)** W. Krull lays foundations for *valuation theory*.

•**(1954)** A. Seidenberg gives a *geometric interpretation for semi-algebraic sets* of Tarski's elimination result.

Today the celebrated *Tarski - transfer principle* is the foundation of modern *semi-algebraic geometry*, the Artin-Schreier theory that of *real algebra*, the work of Hausdorff-Hahn-Hardy that of *ordered algebraic structures and asymptotic analysis.* Combined with the work of Krull, this provides universal *power series constructions of non-Archimedean real closed fields*, i.e. of non-Archimedean models of T.

In his monograph Tarski asks for *analogues results* for $\mathrm{T}_{\exp}$=: $\mathrm{Th}(\mathbb{R}, +, \cdot, <, \exp)$, the elementary theory of the real exponential field.

This led L. van den Dries in the **1980-1990s** to develop *o-minimal geometry*, A. Wilkie in the **1990s** to prove the model completeness and o-minimality of $\mathrm{T}_{\exp}$, and together with A. Macintyre its decidabilty modulo the *real Schanuel conjecture.* The work of Aschenbrenner, van den Dries and van der Hoeven on transseries models culminated in the recent publication of their book. We focused on kappa bounded series constructions of non-Archimedean models of $\mathrm{T}_{an,\exp}$. Inspired by the work of Hardy, we study differential operators on these models. The intriguing relation between *log-atomic monomials* and derivations was recently revealed to us in connection to defining derivations on J. Conway's *field of surreal numbers* $\mathbf{No}$. The flexibility of the construction leads to surprising results. This is the main aspect I want to highlight today.

PART I: Power Series Constructions of Real Closed Fields.

Construction of a Hahn Group:

• Let Γ any totally ordered set and $\mathbb{R}^\Gamma$ the set of all maps g from Γ to $\mathbb{R}$ such that the **support** $\{\gamma \in \Gamma \mid g(\gamma) \neq 0\}$ of g is well-ordered in Γ.

• Endowed with pointwise addition and the lexicographic order, $\mathbb{R}^\Gamma$ is a divisible ordered abelian group **(DOAG)**, called the **Hahn group of rank** Γ.

• $\mathbb{R}^\Gamma$ is archimedean iff Γ is a singleton.

• **Hahn's Embedding Theorem:** a DOAG group of rank Γ is (isomorphic to) a subgroup of $\mathbb{R}^\Gamma$.

Representation for the elements of Hahn groups:

• For every $\gamma \in \Gamma$, we denote by $\mathbf{1}_\gamma$ the map which sends γ to $\mathbf{1}$ and every other element to 0

• $\mathbf{1}_\gamma$ is the characteristic function of the singleton $\{\gamma\}$.

• For $g \in \mathbb{R}^\Gamma$ write

$$g = \sum_{\gamma \in \Gamma} g_\gamma \mathbf{1}_\gamma$$

with $g_\gamma := g(\gamma) \in \mathbb{R}$ and well-ordered support in Γ.

Construction of a Hahn field:

• For $G \neq 0$ an ordered abelian group, $\mathbb{R}((G))$ is the field of generalized **power series** with coefficients in $\mathbb{R}$ and exponents in G:

• A series $s \in \mathbb{R}((G))$ is written

$$s = \sum_{g \in G} s_g t^g$$

with $s_g \in \mathbb{R}$ and well-ordered support $\{g \in G \mid s_g \neq 0\}$.

• As an ordered abelian group, this is just the Hahn group $\mathbb{R}^G$, endowed with convolution (Cauchy) product.

• (Tarski' s recursive axiomatisation:) A totally ordered field K is said to be **real closed** if every positive element has a square root in K, and every polynomial in $K[x]$ of odd degree has a root in K.

• (W. Krull / S. MacLane/ ...) $\mathbb{R}((G))$ is real closed iff G is divisible.

• Let $(K, +, \cdot, 0, 1, <)$ be an ordered field, the **natural valuation** v on K is the valuation whose valuation ring the convex hull of $\mathbb{Z}$ in K.

Notation:

Value group: $G = v(K) := \{v(x) \mid x \in K, x \neq 0\}$.
Valuation ring:, $R_v := \{x \mid x \in K \text{ and } v(x) \geq 0\}$.
Valuation ideal: $I_v := \{x \mid x \in K \text{ and } v(x) > 0\}$.
Group of positive units:
$U_v^{>0} := \{x \mid x \in R_v, x > 0, v(x) = 0\}$.
Residue field: $k = \overline{K} := R_v/I_v$.

Kaplansky Embedding's Theorem (1942): Let K be real closed field with residue field k and value group G. Then K is (isomorphic to) a subfield of a field of $k((G))$.

Direct sum (respectively product) lexicographic decompositions: Let K be real closed field with value group G, then:

$$(K, +, <) = \mathbb{A} \oplus R_v$$

$$(K^{>0}, \cdot, <) = \mathbb{B} \times U_v^{>0}$$

The divisible ordered abelian groups $\mathbb{A}$ (additive) and $\mathbb{B}$ (multiplicative) are unique up to isomorphism. More precisely $\mathbb{B} \simeq G$. The rank of $\mathbb{A}$ is (isomorphic to) $G^{<0}$ as linearly ordered sets.

Example: Let G be a DOAG and set $\mathbb{K} := \mathbb{R}((G))$.

• The natural valuation on $\mathbb{R}((G))$ is $v(s) = \min \operatorname{support} s$ for $s \in \mathbb{R}((G))$, the value group is G.

• The valuation ring $\mathbb{R}((G^{\geq 0}))$ consists of the series with non-negative exponents, the valuation ideal $\mathbb{R}((G^{>0}))$ of the series with positive exponents. The residue field is $\mathbb{R}$

• The **constant term** of a series s is the coefficient s_0. The units of $\mathbb{R}((G^{\geq 0}))$ are the series in $\mathbb{R}((G^{\geq 0}))$ with a non-zero constant term.

• Additive Decomposition:

Given $s \in \mathbb{R}((G))$, we truncate it at its constant term and re-write it as the sum of two series, one with strictly negative exponents, and the other with non-negative exponents. Thus an additive lexicographic complement $\mathbb{A}$ in $(\mathbb{K}, +, <)$ to the valuation ring is the Hahn group $\mathbb{R}^{G^{<0}}$. We call it the **canonical complement to the valuation ring** and denote it by $\mathbb{R}((G^{<0}))$.

• **Multiplicative Decomposition:**

Given $s \in \mathbb{R}((G))^{>0}$, we factor out the monomial of smallest exponent $g \in G$ and re-write $s = t^g u$ as a product, where u a unit with a positive constant term. Thus a multiplicative lexicographic complement $\mathbb{B}$ in $(\mathbb{K}^{>0}, \cdot, <)$ to the subgroup $U_v^{>0}$ of positive units is the group consisting of the (monic) monomials t^g. We call it the **canonical complement to the positive units** and denote it by **Mon** $\mathbb{K}$. Note that, $\mathrm{Mon}\,\mathbb{K}$ is order isomorphic to G through the isomorphism $t^g \mapsto -g$.

PART II: Power Series Constructions of Real Closed Exponential Fields.

A real closed field K is an **exponential field** if there exists a map

$$\exp : (K, +, 0, <) \longrightarrow (K^{>0}, \cdot, 1, <)$$

which is an isomorphism of ordered groups. A map exp with these properties will be called an **exponential** on K. A **logarithm** on K is the compositional inverse $\log = \exp^{-1}$ of an exponential. We always require w.l.o.g. that the exponential be v**-compatible**:

$$\exp(R_v) = U_v^{>0}.$$

We are only interested in exponentials (logarithms) satisfying the **growth axiom** scheme: **(GA)**:

$$\forall n \in \mathbb{N} :\ x > \log(x^n) = n\log(x) \text{ for all } x \in K^{>0} \setminus R_v \ .$$

Via the natural valuation v, this is equivalent to

$$v(x) < v(\log(x)) \text{ for all } x \in K^{>0} \setminus R_v \ . \tag{1}$$

A logarithm log is a **(GA)-logarithm** if it satisfies (1).

Left Exponentiation is imposed on us:

Let $(K, +, \cdot, \exp)$ be a real closed field endowed with a v-compatible exponential exp. We see that exp must restrict to a left exponential, i.e. an isomorphism $\exp : \mathbb{A} \to \mathbb{B}$.

This observation has two mighty consequences:

Theorem 1 [KKS]: Let $G \neq 0$ be a DOAG and set $\mathbb{R}((G))$. Then $\mathbb{R}((G^{<0}))$ is *never* isomorphic to G, so $\mathbb{R}((G))$ cannot admit a v-compatible exponential.

Theorem 2 [S.K.]: Let $(K, +, \cdot, \exp)$ be a real closed field endowed with a v-compatible exponential exp. Let G be its value group and Γ its rank. Then Γ must be isomorphic as linearly ordered set to $G^{<0}$.

Theorem 2 follows immediately from the decomposition theorems.

Call a DOAG G an **exponential group** if its rank Γ is isomorphic as linearly ordered set to $G^{<0}$.

Theorems 1 and 2 are two major obstacles to construction of exponential fields via power series: 1. We cannot use the full field $\mathbb{R}((G))$, we must try to use an appropriate subfield instead 2. We cannot use our favorite DOAG as value group, instead we have to first learn to construct exponential groups.

Characterization of countable exponential groups [S.K]:

A countable DOAG $G \neq 0$ is an exponential group if and only if G is isomorphic to the Hahn sum $\oplus_{\mathbb{Q}} C$ for some countable archimedean group $C \neq 0$.

For the uncountable case we must work much harder.

κ–bounded Hahn Groups and Fields. Fix a regular uncountable cardinal κ.

• The **κ-bounded Hahn group** $(\mathbb{R}^\Gamma)_\kappa \subseteq \mathbb{R}^\Gamma$ consists of all maps of which support has cardinality $< \kappa$.

• The **κ-bounded power series field** $\mathbb{R}((G))_\kappa \subseteq \mathbb{R}((G))$ consists of all series of which support has cardinality $< \kappa$. It is a valued subfield of $\mathbb{R}((G))$. We denote by $\mathbb{R}((G^{\geq 0}))_\kappa$ its valuation ring. Note that $\mathbb{R}((G))_\kappa$ contains the monic monomials.

We denote by $\mathbb{R}((G^{<0}))_\kappa$ the complement to $\mathbb{R}((G^{\geq 0}))_\kappa$.

Proposition 0.1 *Set* $K = \mathbb{R}((G))_\kappa$. *Then* $(K, +, 0, <)$ *decomposes lexicographically as the sum:*

$$(K, +, 0, <) = \mathbb{R}((G^{<0}))_\kappa \oplus \mathbb{R}((G^{\geq 0}))_\kappa \; . \tag{2}$$

$(K^{>0}, \cdot, 1, <)$ *decomposes lexicographically as the product:*

$$(K^{>0}, \cdot, 1, <) = \mathrm{Mon}\,(K) \times U_v^{>0} \tag{3}$$

• **Our goal** is to define an exponential (logarithm) on $\mathbb{R}((G))_\kappa$ (for appropriate choice of G).

Proposition 0.1 allows us to achieve our goal in two main steps; by defining the logarithm on $\mathrm{Mon}\,(K)$ and on $U_v^{>0}$.

Theorem 0.2 *Let* Γ *be a chain, set* $G := (\mathbb{R}^\Gamma)_\kappa$ *and* $K := \mathbb{R}((G))_\kappa$. *Assume that*

$$l : \Gamma \to G^{<0}$$

is an isomorphism of chains. Then l *uniquely defines a logarithm*

$$log \ : (K^{>0}, \cdot, 1, <) \longrightarrow (K, +, 0, <)$$

as follows: given $a \in K^{>0}$, *write* $a = t^g r(1+\varepsilon)$, $g = \sum_{\gamma\in\Gamma} g_\gamma \mathbf{1}_\gamma$, $r \in \mathbb{R}^{>0}$, ε *infinitesimal, and set*

$$\log(a) := -\sum_{\gamma\in\Gamma} g_\gamma t^{l(\gamma)} + \log r + \sum_{i=1}^{\infty} (-1)^{(i-1)} \frac{\varepsilon^i}{i} \qquad (4)$$

This logarithm satisfies

$$v(\log t^g) = l(\min \operatorname{support} g) \tag{5}$$

Moreover, log *satisfies* **GA** *and is a model of* T_{exp} *if and only if*

$$l(\min \operatorname{support} g) > g \qquad \textit{for all } g \in G^{<0}\,. \tag{6}$$

Thus the theorem states that the general necessary condition that the value group be an exponential group is also *sufficient* in this particular context. Therefore our next homework is to construct κ-bounded exponential groups.

The κ-th closure $(\Gamma_\kappa, \iota_\kappa, G_\kappa)$ **of a triplet** (Γ_0, ι_0, G_0)**:**

• **Input:** Γ_0 any non-empty chain, $G_0 = (\mathbb{R}^{\Gamma_0})_\kappa$, and ι_0 a **group section**, that is

$$\iota_0 : \Gamma_0 \to G_0^{<0}$$

is an embedding of linearly ordered sets satisfying

$$\min \operatorname{support}\left(\iota_0(\gamma)\right) = \gamma \text{ for all } \gamma \in \Gamma_0 \, .$$

• **Output:** A chain Γ_κ, the corresponding κ-bounded group $G_\kappa := (\mathbb{R}^{\Gamma_\kappa})_\kappa$, an *isomorphism* of linearly ordered sets

$$\iota_\kappa : \Gamma_\kappa \to G_\kappa^{<0}$$

making G_κ the requested κ-bounded exponential group.

• We shall construct by transfinite induction on $\mu \leq \kappa$ a chain Γ_μ together with an embedding of ordered chains

$$\iota_\mu : \Gamma_\mu \to G_\mu^{<0}$$

where $G_\mu := (\mathbb{R}^{\Gamma_\mu})_\kappa$. We shall have $\Gamma_\nu \subset \Gamma_\mu$ and $\iota_\nu \subset \iota_\mu$ if $\nu < \mu$.

• Assume that for all $\alpha < \mu$ we have already constructed Γ_α, $G_\alpha := (\mathbb{R}^{\Gamma_\alpha})_\kappa$, and the embedding

$$\iota_\alpha : \Gamma_\alpha \to G_\alpha^{<0} \ .$$

First assume that $\mu = \alpha + 1$ is a successor ordinal. Since Γ_α is isomorphic to a subchain of $G_\alpha^{<0}$ through ι_α, we can take $\Gamma_{\alpha+1}$ to be a chain containing Γ_α as a subchain and admitting an isomorphism $\iota_{\alpha+1}$ onto $G_\alpha^{<0}$ which extends ι_α .

More precisely,

$$\Gamma_{\alpha+1} := \Gamma_\alpha \cup (G_\alpha^{<0} \setminus \iota_\alpha(\Gamma_\alpha)) \ ,$$

endowed with the **patch ordering**: if $\gamma_1, \gamma_2 \in \Gamma_{\alpha+1}$ both belong to Γ_α, compare them there, similarly if they both belong to $G_\alpha^{<0}$. If $\gamma_1 \in \Gamma_\alpha$ but $\gamma_2 \in G_\alpha^{<0}$ we set $\gamma_1 < \gamma_2$ if and only if $\iota_\alpha(\gamma_1) < \gamma_2$ in G_α. Then $\iota_{\alpha+1}$ is defined in the obvious way: $\iota_{\alpha+1}|_{\Gamma_\alpha} := \iota_\alpha$ and $\iota_{\alpha+1}|_{(G_\alpha^{<0} \setminus \iota_\alpha(\Gamma_\alpha))} :=$ the identity map. Note that

$$\iota_{\alpha+1}(\Gamma_{\alpha+1}) = G_\alpha^{<0}. \tag{7}$$

Thus $\iota_{\alpha+1}$ is an embedding of $\Gamma_{\alpha+1}$ into $G_{\alpha+1}^{<0}$.

If μ is a limit ordinal we set

$$\Gamma_\mu := \bigcup_{\alpha<\mu} \Gamma_\alpha \ , \ \ \iota_\mu := \bigcup_{\alpha<\mu} \iota_\alpha \quad \text{and} \quad G_\mu := (\mathbb{R}^{\Gamma_\mu})_\kappa \ .$$

Note that by construction and (7)

$$\iota_\mu(\Gamma_\mu) = \bigcup_{\alpha<\mu} G_\alpha^{<0} \tag{8}$$

and $\bigcup_{\alpha<\mu} G_\alpha \subset G_\mu$.
This completes the construction of $\Gamma_\kappa := \bigcup_{\alpha<\kappa} \Gamma_\alpha$, $\iota_\kappa := \bigcup_{\alpha<\kappa} \iota_\alpha$ and $G_\kappa := (\mathbb{R}^{\Gamma_\kappa})_\kappa$.

We now claim that

$$G_\kappa = \bigcup_{\alpha<\kappa} G_\alpha$$

Once the claim is established, we conclude from (8) that $\iota_\kappa : \Gamma_\kappa \to G_\kappa^{<0}$ is an isomorphism, as required. Let $g \in G_\kappa$ and $\kappa > \delta := \mathrm{card}(\mathrm{support}\, g)$. Now $\mathrm{support}\, g := \{\gamma_\mu \,;\, \mu < \delta\} \subset \Gamma_\kappa$, so for every $\mu < \delta$ choose $\alpha_\mu < \kappa$ such that $\gamma_\mu \in \Gamma_{\alpha_\mu}$. Clearly $\mathrm{card}(\{\alpha_\mu \,;\, \mu < \delta\}) \leq \delta < \kappa$ so $\{\alpha_\mu\,; \mu < \delta\}$ cannot be cofinal in κ since κ is **regular**, therefore it is bounded above by some $\alpha \in \kappa$. It follows that $\mathrm{support}\, g \subset \Gamma_\alpha$, so $g \in G_\alpha$ as required.

Finally we take care of **(GA)** by exploiting **right shifts** of Γ_0 to improve the group exponential ι_κ:

Proposition 0.3 *Assume that* $\sigma_\kappa \in \operatorname{Aut}(\Gamma_\kappa)$ *is such that* $\sigma_\kappa|_{\Gamma_\mu} \in \operatorname{Aut}(\Gamma_\mu)$ *for all* $\mu \in \kappa$ *and* $\sigma_\kappa(\gamma) > \gamma$ *for all* $\gamma \in \Gamma_0$. *Then the isomorphism*

$$l := \iota_\kappa \circ \sigma_\kappa : \Gamma_\kappa \to G_\kappa^{<0}$$

satisfies (6).

We are now ready to summarize the procedure of constructing the **Exponential-Logarithmic field of κ-bounded series over** $(\Gamma_0, \iota_0, G_0, \sigma_0)$ where σ_0 a right shift automorphism of Γ_0.

• Construct $\Gamma_\kappa, \quad G_\kappa, \quad \iota_\kappa, \quad$ and σ_κ.

• Set $K := \mathbb{R}((G_\kappa))_\kappa$ and $l := \iota_\kappa \circ \sigma_\kappa$. Note that l is surjective and satisfies (6).

• Denote by log the surjective **GA** logarithm induced on $K^{>0}$ by l and set $\exp = \log^{-1}$.

• $(K, \exp)$ is a model of $T_{\mathrm{an,exp}}$.

PART III: Group sections and log-atomic monomials.

Set K:= EL$(\kappa, \Gamma_0, \iota_0, G_0, \sigma_0)$.

- A monomial t^g is **log-atomic** if $\log_n(t^g)$ is a monomial for all $n \in \mathbb{N}$.
- Log-atomic elements are fundamental when defining series derivations on K.
- Group sections always exist and can vary widely.
- It turns out that the set of log-atomic monomials depends on the choice of the group section.

For the computations below recall that

$$\log(t^g) = \sum_{\gamma \in \Gamma} -g_\gamma t^{l(\gamma)}$$

for $g = \sum_{\gamma \in \Gamma} g_\gamma 1_\gamma$. In particular $\log(t^{-\mathbf{1}_\gamma}) = t^{l(\gamma)}$.

• We will need the following simple observation:

A necessary condition for log-atomic: Assume that t^g is log-atomic and set $\log_n(t^g) := t^{g_n}$. Then g_n must have singleton support for all $n \in \mathbb{N}_0$ (i.e. t^{g_n} is a **fundamental monomial**).

Below denote by $\sigma^n(\gamma)$ the **n-th iterate** of $\sigma := \sigma_0$.

The basic section: is defined by $\gamma \mapsto -\mathbf{1}_\gamma$. The corresponding $l : \Gamma \to G_0^{<0}$ is defined by $\gamma \mapsto -\mathbf{1}_{\sigma(\gamma)}$. By induction

$$\log{}_n(t^{-\mathbf{1}_\gamma}) = t^{-\mathbf{1}_{\sigma^n(\gamma)}} \; .$$

So $t^{-\mathbf{1}_\gamma}$ is log-atomic for all $\gamma \in \Gamma_0$.

We will see later that in this case, we can define a derivation.

The 2-element support section:

• Since Γ_0 has no last element, for every $\gamma \in \Gamma_0$ let $\tau(\gamma) > \gamma$. Consider the section $\gamma \mapsto -\mathbf{1}_\gamma + \mathbf{1}_{\tau(\gamma)}$.

• We claim that the corresponding log has no log-atomic elements.

• We prove this for t^g with $g \in G_\kappa = \bigcup_{\alpha<\kappa} G_\alpha$, by transfinite induction on α.

• $\alpha = 0$: Let $g \in G_0$ have singleton support, say $g = -\mathbf{1}_\gamma$. Compute $\log(t^g) = t^{l(\gamma)}$. But $l(\gamma) = -\mathbf{1}_{\sigma(\gamma)} + \mathbf{1}_{\tau(\sigma(\gamma))}$. So t^g is not log-atomic.

• α is a limit ordinal: clear by induction hypothesis.

• Consider now $g \in G_{\alpha+1}$. As before g has singleton support $\gamma \in \Gamma_{\alpha+1}$, say $g = -\mathbf{1}_\gamma$, compute $\log(t^g) = t^{l(\gamma)}$. But $l(\gamma) \in G_\alpha^{<0}$ by (7), so $\log(t^g) = t^{l(\gamma)}$ is not log-atomic, a fortiori t^g is not log-atomic.

We will see later that in this case we cannot define a derivation.

PART IV: Growth rates and the T-convex rank.

• Let Γ be a chain and $\sigma \in \operatorname{Aut}(\Gamma)$ an increasing automorphism. By induction, we define the **n-th iterate** of σ: $\sigma^1(\gamma) := \sigma(\gamma)$ and $\sigma^{n+1}(\gamma) := \sigma(\sigma^n(\gamma))$. Define an equivalence relation on Γ as follows: For $\gamma, \gamma' \in \Gamma$, set

$$\gamma \sim_\sigma \gamma' \text{ iff } \exists n \in \mathbb{N} \text{ s.t. } \sigma^n(\gamma) \geq \gamma' \text{ and } \sigma^n(\gamma') \geq \gamma \,.$$

The equivalence classes $[\gamma]_\sigma$ of $\sim_\sigma$ are convex and closed under application of σ (they are the convex hulls of the orbits of σ). The order of Γ induces an order on $\Gamma/\sim_\sigma$. The order type of $\Gamma/\sim_\sigma$ is the **rank** of (Γ, σ).

Example 0.4 Let $\Gamma = \mathbb{Z} \vec{\amalg} \mathbb{Z}$ (i.e. the lexicographically ordered Cartesian product $\mathbb{Z} \times \mathbb{Z}$) endowed with the automorphism $\sigma((x, y)) := (x, y + 1)$. The rank of σ is $\mathbb{Z}$. Now consider the increasing automorphism
$\tau((x, y)) := (x + 1, y)$. The rank of τ is 1.

• Let K be a real closed field and log a (**GA**)- logarithm on $K^{>0}$. Define an equivalence relation on $K^{>0} \setminus R_v$:

$$a \sim_{log} a' \text{ iff } \exists n \in \mathbb{N} \text{ s.t. } \log_n(a) \leq (a') \text{ and } \log_n(a') \leq a$$

(where $\log_n$ is the n-th iterate of the log). The order type of the chain of equivalence classes is the **logarithmic rank** of $(K^{>0}, \log)$.

We can compute the logarithmic rank of the Exponential-Logarithmic field of κ-bounded series over (Γ, σ):

Theorem 0.5 *The logarithmic rank of* $(\mathbb{R}((G_\kappa))_\kappa^{>0}, \log)$ *is equal to the rank of* (Γ, σ).

The logarithmic rank parametrizes the (inclusion) linearly ordered set of those convex valuation rings that are exp and log closed. Since these are models of $T := T_{\exp}$*, which is exponentially bounded, it follows that the logarithmic rank parametrizes the (inclusion) linearly ordered set of those T- convex valuation rings (in the sense of vDD).*

This proof (as many other proofs) is based on the observation that every series is log-equivalent to a **fundamental monomial**, that is a monomial of the form

$$t^{-\mathbf{1}_\gamma} \text{ with } \gamma \in \Gamma \,.$$

Next one observes that

$$\text{for all } \gamma, \gamma' \in \Gamma : t^{-\mathbf{1}_\gamma} \sim_{log} t^{-\mathbf{1}_{\gamma'}} \text{ if and only if } \gamma \sim_\sigma \gamma' \,.$$

This in turn is based on the following useful formula for $\log_n(t^{-\mathbf{1}_\gamma})$: by induction,

$$\log{}_n(t^{-\mathbf{1}_\gamma}) = t^{-\mathbf{1}_{\sigma^n(\gamma)}} \,.$$

PART V: Chains with many automorphisms and transexponentials.

If Γ admits automorphisms of distinct rank, then $(\mathbb{R}((G_\kappa))$ admits logarithms of distinct logarithmic rank. We can also use this observation to introduce **transexponentials**, as illustrated in the next example.

Example 0.6 Let $\Gamma = \mathbb{Z} \vec{\amalg} \mathbb{Z}$, $\sigma((x,y)) := (x, y+1)$, $(K, \log)$ the corresponding κ-bounded model. For the automorphism $\tau((x,y)) := (x+1, y)$, let L, respectively $T := L^{-1}$ be the corresponding induced logarithm and exponential on K.

Effect of σ, τ on the fundamental monomials: let $\gamma = (x,y) \in \Gamma$, then

$$\log(t^{-\mathbf{1}_\gamma}) = t^{-\mathbf{1}_{\sigma(\gamma)}} ,$$

Whereas

$$L(t^{-\mathbf{1}_\gamma}) = t^{-\mathbf{1}_{\tau(\gamma)}} ,$$

We see that, for any fundamental monomial $X := t^{-\mathbf{1}_\gamma}$ and any $n \in \mathbb{N}$ we have:

$$L(X) < \log_n(X) .$$

Also, a simple computation (using the fact that σ and τ commute) shows that also, for all $n \in \mathbb{N}$:

$$T(X) > \exp_n(X) .$$

In the next part, we see how the logarithm determines the derivation. We expect to obtain fields equipped with several distinct derivations.

PART VI: Derivations.

Main motivation: We want a "Kaplansky embedding Theorem" for ordered differential fields. The κ-bounded fields of power series are good candidates as "universal domains". But for this to make sense, we need first to endow them with a good differential structure.

Main task: Given (Γ, σ), introduce, if possible, derivation and composition operators on Exponential-Logarithmic field of κ-bounded series over (Γ, σ).

We want to endow the (pre)logarithmic field of κ-bounded series over (Γ, σ) with a derivation D satisfying the following properties:

• D is strongly linear, that is

$$D \sum_g r_g t^g \; = \; \sum_g r_g D t^g \; . \tag{9}$$

• D satisfies Leibniz rule:

$$D(ab) = aD(b) + D(a)b \tag{10}$$

• D satisfies the rule for the logarithmic derivative for $a > 0$:

$$D \log a = Da/a \tag{11}$$

Reductions: The above rules direct us to perform a number of steps in "trying" (that is, modulo **summability issues!**) to define derivatives:

(i) From (9) and (10), it is clear that we only need to determine Dt^g, for $g \in G^{<0}$.

(ii) From (11) determining Dt^g reduces to determining $D \log t^g$.

(iii) By definition of log, this in turn reduces to determining $D \log t^{-\mathbf{1}_\gamma}$, for a fundamental monomial $t^{-\mathbf{1}_\gamma}$ with $\gamma \in \Gamma$.

For the basic section:

(iv) Applying (11) again we see that for any $\gamma \in \Gamma_0$ we have:

$$Dt^{-\mathbf{1}_{\sigma(\gamma)}} = t^{\mathbf{1}_\gamma} Dt^{-\mathbf{1}_\gamma} .$$

(v) we obtain from (iv), we see that we only need to define $Dt^{-\mathbf{1}_{\gamma_0}}$ for a fixed representative $\gamma_0 \in \Gamma$ of an orbit of σ in Γ.

Example 0.7 Let $\Gamma = \mathbb{Z}$ endowed with the basic section and the automorphism $\sigma(z) := z + 1$. For simplicity, let us choose $\gamma_0 = 0$ and set

$$T := t^{-\mathbf{1}_0} \text{ and } DT = 1\,.$$

Then $t^{-\mathbf{1}_n} = \log_n T$ if $n > 0$, and $t^{-\mathbf{1}_n} = \exp_{-n} T$ if $n < 0$. Therefore, for $n > 0$

$$Dt^{-\mathbf{1}_n} = \prod_{k=0}^{n-1} t^{\mathbf{1}_k} \quad \text{and} \quad Dt^{-\mathbf{1}_{-n}} = \prod_{k=1}^{n} t^{-\mathbf{1}_{-k}}\,.$$

It is non-trivial to verify that these definitions induce a *well-defined* derivative, that is, to verify **summability**.

Same example as above but instead with the 2 element section fails to admit derivations, namely the family $log(t^{-1_n}) = t^{-1_{n+1}+1_{n+2}}$ *is summable but not that of the derivatives.*

Example 0.8 Let $\Gamma = \mathbb{Z}\,\vec{\amalg}\,\mathbb{Z}$ endowed with the automorphism $\sigma((x,y)) := (x, y+1)$. The rank of σ is $\mathbb{Z}$. For each orbit of σ_0 we fix a representative $z \in \mathbb{Z}$. We set $T_z := t^{-\mathbf{1}_z}$. Then $\{T_z\ ;\ z \in \mathbb{Z}\}$ will represent infinitely many algebraically independent variables, which will determine an infinite family $\{\delta_z\}$ of commuting partial derivatives.

What about a derivation induced by the automorphism $\tau((x,y)) := (x+1, y)$ of rank one? This is more challenging. We have countably many distinct orbits but with a single common convex hull. This suggests defining " arbitrary iterates" $\log_\gamma T$ of the log, to capture the derivative of every fundamental monomial.