Convex valuations on ordered fields, with particular emphasis on fields of generalised power series

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Caserta, April 05’ 2011

Part I: Convex valuations on ordered fields

§1. Ordered abelian groups

Let $(\Gamma, +, 0, \leq)$ be an ordered abelian group written additively.
i.e. it satisfy axioms of **total order**:

(1) $\gamma \leq \gamma$ (Reflexive)
(2) $\gamma \leq \delta$, $\delta \leq \Gamma \Rightarrow \Gamma = \delta$ (Antisymmetric)
(3) $\Gamma \leq \delta$, $\delta \leq \lambda \Rightarrow \Gamma \leq \lambda$ (Transitive)
(4) $\Gamma \leq \delta$ or $\delta \leq \Gamma$ (Total)
(5) Compatible with $+$: $\Gamma \leq \delta \Rightarrow \Gamma + \lambda \leq \delta + \lambda$.

Definition 1.1. Convex Subgroups: $\Delta \leq \Gamma$ convex if $\forall\ \delta \in \Delta, \gamma \in \Gamma$ with $0 \leq \gamma \leq \delta : \gamma \in \Delta$.
(Note: Torsion free: $\gamma > 0 \Rightarrow \gamma < 2\gamma < \dots$)

Definition 1.2. The collection of $\{\Delta \subsetneq \Gamma; \Delta$ convex proper subgroup $\}$ is **totally ordered** by inclusion.
The order type of this ordered set is called the **rank** of Γ.
e.g. $\{0\}$ is a convex subgroup. (Rank 1 valuations)

Thus if Γ has exactly n proper convex subgroups, we say that Γ has rank n, where $n \in \mathbb{N}_+ = \{1, 2, \ldots\}$.

e.g. if $\{0\}$ is the only convex subgroup say Γ has rank 1.

e.g. $\mathbb{Z}$ has rank 1 (i.e. to show that if $\Delta \neq 0$, Δ convex $\Rightarrow \Delta = \mathbb{Z}$)
(discrete rank 1 valuations)

Rank 1 is characterised by the archimedean property:
$\forall \gamma, \epsilon \in \Gamma$ such that $\epsilon > 0\ \exists\, n \in \mathbb{N}$ s.t. $-\gamma, \gamma \leq n\epsilon$.

Example: $(\mathbb{Z}, +), (\mathbb{Q}, +), (\mathbb{R}, +), (\mathbb{R}^{>0}, .)$ all are archimedean.

e.g. (higher rank) $\underbrace{\mathbb{Z}_{(1)} \times \mathbb{Z}_{(2)} \ldots \times \mathbb{Z}_{(n)}}_{\text{(the direct product endowed with lexicographic order)}}$ has rank n,

proper convex subgroups are:

$\mathbb{Z}_{(n)}$

$\mathbb{Z}_{(n-1)} \times \mathbb{Z}_{(n)}$

$\mathbb{Z}_{(n-2)} \times \mathbb{Z}_{(n-1)} \times \mathbb{Z}_{(n)}$

doing it for $n = 2$, $\mathbb{Z} \times \mathbb{Z}$ has 2 proper convex subgroups:
$\Delta_1 = 0$, $\Delta_2 =$ second copy of $\mathbb{Z} = \{(0, z) \mid z \in \mathbb{Z}\}$
[Since $(0, 0) \leq (z_1, z_2) \leq (0, z)$]

Lemma 1.3. Γ is archimedean iff $\mathrm{rank}(\Gamma) = 1$

Proof. "$\Rightarrow$" Assume Γ archimedean, $\Delta \neq 0, \Delta$ convex **show** $\Delta = \Gamma$
fix $\delta > 0$; $\delta \in \Delta$ and $\gamma \in \Gamma$, wlog $\gamma > 0$.
By the archimedean property $\exists\, n$ s.t. $0 < \gamma < n\delta$
then by convexity $\gamma \in \Delta$. □

"$\Leftarrow$" Assume only $\{0\}$ is convex, **show** Γ archimedean
Fix $\epsilon \in \Gamma$; $\epsilon > 0$ we want to **prove that** $\forall \gamma \in \Gamma\ \exists\, n \in \mathbb{N}$ s.t. $-\gamma, \gamma \leq n\epsilon$

Set $\Delta := \{\gamma \in \Gamma\ ;\ -\gamma, \gamma \leq n\epsilon$ for some $n \in \mathbb{N}\}$

Clearly $0 \in \Delta$, $\gamma \in \Delta \Rightarrow -\gamma \in \Delta$
Also $\gamma_1, \gamma_2 \in \Delta \Rightarrow -\gamma_1, \gamma_1 \leq n_1\epsilon$; $-\gamma_2, \gamma_2 \leq n_2\epsilon$
$\Rightarrow -(\gamma_1 + \gamma_2), (\gamma_1 + \gamma_2) \leq (n_1 + n_2)\epsilon$
So, Δ is a subgroup.

Δ is convex: since for $\gamma \in \Gamma,\ 0 \leq \gamma \leq \delta \in \Delta \Rightarrow \gamma \in \Delta$

$\Delta \neq \{0\}$, since $\epsilon \in \Delta$

So $\Delta = \Gamma$ and Γ is archimedean. □

Theorem 1.4. (Hölder) Γ is archimedean $\Leftrightarrow$ isomorphic to a subgroup of $(\mathbb{R}, +, 0, \leq)$.

Proof. Assume $\Gamma \neq \{0\}$
Fix $\epsilon \in \Gamma$; $\epsilon > 0$
for any $\gamma \in \Gamma$ consider

$$L(\gamma) := \left\{ \tfrac{m}{n} \in \mathbb{Q} \mid (n > 0) \text{ and } m\epsilon \leq n\gamma \right\}$$

$$U(\gamma) := \left\{ \tfrac{m}{n} \in \mathbb{Q} \mid (n > 0) \text{ and } m\epsilon \geq n\gamma \right\}$$

Show $L(\gamma) \neq \phi$, $U(\gamma) \neq \phi$, $L(\gamma) \leq U(\gamma)$, $L(\gamma) \cup U(\gamma) = \mathbb{Q}$

Dedekind cut in the rationals
$\gamma \longmapsto r(\gamma)$,
where $r(\gamma)$ is the real determined by the Dedikind cut $(L(\gamma), U(\gamma))$. □

Example 1.5 The direct product $\mathbb{Z}\times\mathbb{Z}$ is discrete (has a smallest positive element) of rank 2, when endowed with the lexicographic order.

We can endow it with ordering of rank 1,
namely $\mathbb{Z} \times \mathbb{Z}$ is identified with the (additive) subgroup $\mathbb{Z} + \mathbb{Z}\sqrt{2}$ of $(\mathbb{R}, +, 0, \leq)$.
With this ordering $\mathbb{Z}\times\mathbb{Z}$ is archimedean and densely ordered ($\gamma_1 < \gamma_2 \Rightarrow \exists\, \gamma_3$ s.t. $\gamma_1 < \gamma_3 < \gamma_2$).

§2. Valued fields

Let $\infty > \Gamma$, K a field
$v : K \twoheadrightarrow \Gamma \cup \{\infty\}$, then

(1) $v(x) = \infty \Leftrightarrow x = 0$

(2) $v(xy) = v(x) + v(y)$

(3) $v(x + y) \geq \min\{v(x), v(y)\}$

Proposition 2.1. (Basic properties:)
(4) $v(1) = 0$ and $v(x) = v(-x), x \neq 0$

(5) for $x \neq 0$, $v(x^{-1}) = -v(x)$

(6) for $y \neq 0$, $v\left(\frac{x}{y}\right) = v(x) - v(y)$

(7) $v(x) < v(y) \Rightarrow v(x + y) = v(x)$

Proof of (7). Assume for a contradiction that
$v(x + y) > v(x)$ and compute:

$$\begin{aligned} v(x) = v\Big((x + y) - x\Big) &\geq \min\{v(x + y), v(-y)\} \\ &= \min\{v(x + y), v(y)\} \\ &> v(x)\,, \text{ a contradiction.} \end{aligned}$$

□

Definition 2.2. $\mathcal{O}_v := \{x \in K \mid v(x) \geq 0\}$ is a **valuation ring** of K, i.e. it satisfies that $\forall\ x \in K^\times : x \in \mathcal{O}_v$ or $x^{-1} \in \mathcal{O}_v$.

Definition 2.3. The **group of units** of $\mathcal{O}_v$ is

$$\mathcal{O}_v^\times := \{x \in K \mid x, x^{-1} \in \mathcal{O}_v\} = \{x \in K \mid v(x) = 0\}.$$

Definition 2.4. The **set of non units** of $\mathcal{O}_v$ is

$$\begin{aligned} \mathfrak{m}_v &:= \{x \in K \mid v(x) \geq 0 \text{ but } v(x) \neq 0\} \\ &= \{x \in K \mid v(x) > 0\}, \end{aligned}$$

is an ideal; it is a maximal ideal, and the **unique maximal ideal** [Since I ideal, $I \supsetneq \mathfrak{m}_v \Rightarrow I$ contains a unit of $\mathcal{O}_v \Rightarrow I = \mathcal{O}_v$] (**proper** since $v(1) = 0$).
(So that $\mathcal{O}_v$ is a so called "local ring").

Definition 2.5. $\overline{K_v} := \mathcal{O}_v/\mathfrak{m}_v$ is a field called the **residue field**.
The canonical homomorphism

$$\begin{aligned} \mathcal{O}_v &\twoheadrightarrow \overline{K_v} \\ x &\longmapsto x + \mathfrak{m}_v \end{aligned}$$

is the residue map.

So, $\overline{x} := x + \mathfrak{m}_v$ is zero $\Leftrightarrow x \in \mathfrak{m}_v$, nonzero $\Leftrightarrow x \in \mathcal{O}_v^\times$.

Example 2.6. Let k be any field and consider
$k[X]$:= polynomial ring in 1-variable,

$K := k(X) := qq(k[X])$ = rational function field in 1-variable.

The **degree valuation** $v := -\deg$ on K is defined by

$$v : K \twoheadrightarrow \mathbb{Z} \cup \{\infty\}$$

$$v\Big(\frac{f}{g}\Big) := \deg g - \deg f$$

The axioms can be easily verified. Also,

Valuation ring $\mathcal{O}_v := \left\{\frac{f}{g} \in K \mid \deg g \geq \deg f\right\}$

Maximal ideal $\mathfrak{m}_v := \left\{\frac{f}{g} \in K \mid \deg g > \deg f\right\}$

Units $\frac{f}{g}$ is a unit $\Leftrightarrow \deg g = \deg f$

Residues If $f(X) \in k[X]$,
$f(X) = a_n X^n + \ldots + a_0$; $a_n \neq 0, a_i \in k$
then
$\mathfrak{u} := \frac{f(X)}{X^n}$ is a unit of $\mathcal{O}_v$

Let us compute $\underline{\mathfrak{u}}$?

We **claim** that $\overline{\mathfrak{u}} = a_n$, i.e. we show that $\mathfrak{u} - a_n \in \mathfrak{m}_v$:
Now

$$\mathfrak{u} = a_n + \tfrac{a_{n-1}}{X} + \tfrac{a_{n-2}}{X^2} + \ldots + \tfrac{a_0}{X^n}$$

$$\Rightarrow \mathfrak{u} - a_n = \underbrace{\frac{a_{n-1}}{X}}_{\in \mathfrak{m}_v} + \underbrace{\frac{a_{n-2}}{X^2}}_{\in \mathfrak{m}_v} + \ldots + \underbrace{\frac{a_0}{X^n}}_{\in \mathfrak{m}_v}$$

$$\in \mathfrak{m}_v \text{ (Since } \mathfrak{m}_v \text{ is an ideal)}$$

So residue field is k. □

§3. Ordered fields - Real closed fields

Definition 3.1. Totally ordered fields: $(K, +, ., 0, 1, \leq)$ is an ordered field if $(K, +, 0, \leq)$ is an ordered abelain group and compatible with multiplication ($x \leq y \Rightarrow zx \leq zy$ if $z \geq 0$).
It follows: $1 > 0$, $-1 < 0$, $x^2 \geq 0$, -1 is not a square.
$\Rightarrow \mathrm{Char} K = 0$

$\mathbb{C}$ admits no ordering.

Analogue of ``algebraically closed fields '' for class of ordered fields is Real closed fields.

Theorem 3.2. (Artin Schreier) Let $(K, \leq)$ be an ordered field, then TFAE:

(i) $(K, \leq)$ has no proper ordered algebraic extension.

(ii) in $(K, \leq)$ every positive element is a square and every odd degree polynomial $f \in K[X]$ has a zero in K.

(iii) $K\left(\sqrt{-1}\right)$ is algebraically closed and $K \neq K\left(\sqrt{-1}\right)$.

(iv) $[\tilde{\mathrm{K}}^{\mathrm{alg}} : K] = 2$.

Any such ordered field is a RCF.

Examples 3.3.

- **Examples of RCF:**
  (i) $\mathbb{Q}^{\text{ralg}}$: real algebraic numbers.
  (ii) $\mathbb{R}$ with its ordering ($r > 0 \Rightarrow r = s^2$ and IVT).
  [More by power series constructions.
  k real closed, Γ divisible ordered abelian group $\Rightarrow k((\Gamma))$ real closed.]
- **Examples of ordered fields (not necessarily real closed):**
  (i) $\mathbb{Q}$ $\left(\sqrt{2} \notin \mathbb{Q}\right)$
  (ii) $\mathbb{R}$
  These are Archimedean fields (archimedean property) of the reals.
  By Hölder: every such field is a subfield of the reals.

Are there non archimedean ordered fields?

Well since $\mathbb{R}$ is real closed by the fact (theorem above) we cannot produce algebraic examples so let us go to trascendental examples:

$\mathbb{R}(t)$ = Rational function field in one variable
$\mathbb{R}(t) := qf(\mathbb{R}[t])$
$f(X) = a_0 + \ldots + a_n X^n; a_i \in \mathbb{R}$

Decide on the sign of f by looking at the sign of the lowest coefficient:

X and X^2 are both positive but also

$X - nX^2$ is positive for all $n \in \mathbb{N}$, i.e. $X - nX^2 > 0\ \forall\ n \in \mathbb{N}$

So, $X > nX^2\ \forall\ n$

So, $X >> X^2$

Also for $a \neq 0, a \in \mathbb{R}$:

$a - nX > 0\ \forall\ n \in \mathbb{N}$

i.e. $a > nX \; \forall \; n$

So, $X << \mathbb{R}$ $\square$

More examples with power series fields in lecture II. Measure the degree of ``nonachimedeanity''.

§4. The natural valuation in an ordered field

Definition 4.1. The natural valuation has a convex valuation ring, in fact the valuation ring is the convex hull of $\mathbb{Z}$, this is the ring of finite elements.

- $v(1) = 0$
- $v(a) \geq 0$ i.e. $v(a) \geq v(1)$
  either $\underbrace{a \sim^{+} 1}_{\text{(units)}}$ or $\underbrace{a <^{+}< 1}_{\text{(non units)}}$ (ideal of infinitesimals).
- $v(0) = \infty$
  $\infty > v(K)$

Compatibility:
$0 < a < b \Rightarrow v(b) < v(a)$ $\square$

Part II: Fields of generalized power series