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The Moment problem for the real polynomial algebra in infinitely many variables

The univariate moment problem

Is an old problem with origins tracing back to work of Stieltjes. Given a sequence $(s_k)_{k\geq 0}$ of real numbers one wants to know when there exists a Radon measure μ on $\mathbb{R}$ such that

$$s_k = \int x^k d\mu \; \forall \, k \geq 0.^1$$

Since the monomials x^k, $k \geq 0$ form a basis for the polynomial algebra $\mathbb{R}[x]$, this problem is equivalent to the following one: Given a linear functional $L : \mathbb{R}[x] \to \mathbb{R}$, when does there exist a Radon measure μ on $\mathbb{R}$ such that $L(f) = \int f d\mu \; \forall \, f \in \mathbb{R}[x]$. One also wants to know to what extent the measure is unique, assuming it exists. Akhiezer 1965 and Shohat-Tamarkin 1943 are standard references.

[1] All Radon measures considered are assumed to be positive.

THE MULTIVARIATE MOMENT PROBLEM

Has been considered more recently. For $n \geq 1$, $\mathbb{R}[\underline{x}] := \mathbb{R}[x_1, \ldots, x_n]$ denotes the polynomial ring in n variables $x_1, \ldots, x_n$. Given a linear functional $L : \mathbb{R}[\underline{x}] \to \mathbb{R}$ and a closed subset Y of $\mathbb{R}^n$ one wants to know when there exists a Radon measure μ on $\mathbb{R}^n$ supported on Y such that $L(f) = \int f d\mu \ \forall f \in \mathbb{R}[\underline{x}]$.

Haviland, 1936

Such a measure exists if and only if $L(\text{Pos}(Y)) \subseteq [0, \infty)$, where $\text{Pos}(Y) := \{f \in \mathbb{R}[\underline{x}] : f(x) \geq 0 \quad \forall x \in Y\}$.

Again, one also wants to know to what extent the measure is unique, assuming it exists. Berg 1987, Fuglede 1983 are general references. A major motivation here is the close connection between the multivariate moment problem and real algebraic geometry; see e.g. Schmüdgen 1999, Marshall 2008, Lasserre 2013.

The infinite-variate moment problem

There is work dealing with the moment problem in infinitely many variables, mainly where the linear functional in question is continuous for a certain topology. Albeverio-Herzberg 2008 applies Schmüdgen's 1999 solution of the moment problem to represent L^1-continuous linear functionals on the vector space of polynomials of Brownian motion as integration with respect to probability measures on the Wiener space of $\mathbb{R}$.
Berezansky-Kondratiev 1995, Berezansky-Sifrin 1971, Borchers-Yngvason 1975, Hegerfeldt 1975, Infusino-Kuna-Rota 2014, Infusino 2015 consider continuous linear functionals on the symmetric algebra of a nuclear space.

Ghasemi-Kuhlmann-Marshall 2014 prove a general integral representation theorem for positive continuous linear functionals on a locally multiplicatively convex (lmc) topological real algebra.
Ghasemi-Infusino-Kuhlmann-Marshall 2018 applies the 2014 result to linear functionals on the symmetric algebra of a locally convex space (V, τ) which are continuous with respect to the finest lmc topology extending τ. In today's paper, we deal with the general question for the algebra of polynomials in an arbitrary number of variables, and without continuity assumptions on the positive functionals under consideration.
Today, I want to focus on the following result

Extension of Haviland's theorem

Let $A = A_\Omega := \mathbb{R}[x_i \mid i \in \Omega]$, the ring of polynomials in an arbitrary number of variables x_i, $i \in \Omega$ with coefficients in $\mathbb{R}$.

Extension of Haviland

Suppose $L : A_\Omega \to \mathbb{R}$ is linear and $L(\mathrm{Pos}_{A_\Omega}(Y)) \subseteq [0, \infty)$ where Y is a closed subset of $\mathbb{R}^\Omega$ satisfying condition (i) below. Then there exists a constructibly Radon measure ν on $\mathbb{R}^\Omega$ supported by Y such that $L(f) = \int \hat{f} d\nu \; \forall f \in A_\Omega$.

Condition (i): Y is described by countably many inequalities i.e., there exists a countable $S \subset A_\Omega$ such that $Y = \{\alpha \in \mathbb{R}^\Omega \mid \hat{g}(\alpha) \geq 0 \; \forall g \in S\}$. We note that Condition (i) is always satisfied for countable Ω.

Extension of Haviland in the countable case

Suppose Ω is countable, $L : A_\Omega \to \mathbb{R}$ is linear and $L(\mathrm{Pos}_{A_\Omega}(Y)) \subseteq [0, \infty)$ where Y is a closed subset of $\mathbb{R}^\Omega$. Then there exists a Radon measure ν on $\mathbb{R}^\Omega$ supported by Y such that $L(f) = \int \hat{f} d\nu \; \forall f \in A_\Omega$.

Terminology, notations, general setting

- All rings considered are commutative with 1.
- All ring homomorphisms considered send 1 to 1.
- All rings we are interested in are $\mathbb{R}$-algebras.
- For a commutative ring A, $X(A)$ the character space of A is the set of all ring homomorphisms $\alpha : A \to \mathbb{R}$, .
- For $a \in A$, $\hat{a} = \hat{a}_A : X(A) \to \mathbb{R}$ is defined by $\hat{a}_A(\alpha) = \alpha(a)$.
- $X(A)$ is given the weakest topology such that the functions $\hat{a}_A$, $a \in A$ are continuous.
- The only ring homomorphism from $\mathbb{R}$ to itself is Id.
- Ring homomorphisms from $\mathbb{R}[\underline{x}]$ to $\mathbb{R}$ correspond to point evaluations $f \mapsto f(\alpha)$, $\alpha \in \mathbb{R}^n$. $X(\mathbb{R}[\underline{x}])$ is identified as a topological space with $\mathbb{R}^n$.

- A quadratic module of A is a subset M of A satisfying

$$1 \in M,\ M + M \subseteq M \text{ and } a^2 M \subseteq M \text{ for each } a \in A.$$

- A quadratic preordering of A is a quadratic module of A which is also closed under multiplication.
- For a subset X of $X(A)$,

$$\mathrm{Pos}_A(X) := \{a \in A \mid \hat{a}_A \geq 0 \text{ on } X\}$$

  is a preordering of A.
- $\sum A^2$ the set of all finite sums $\sum a_i^2$, $a_i \in A$. It is the unique smallest quadratic module (preordering) of A.
- For a subset $S \subseteq A$,

$$X_S := \{\alpha \in X(A) \mid \hat{a}_A(\alpha) \geq 0\ \forall a \in S\}.$$

- A quadratic module M in A is archimedean if for each $a \in A$ there exists an integer k such that $k \pm a \in M$.
- If M is a quadratic module of A which is archimedean then X_M is compact.

Archimedean Positivstellensatz

Suppose M is an archimedean quadratic module of A. Then, for any $a \in A$, the following are equivalent:

(1) $\hat{a}_A \geq 0$ on X_M.

(2) $a + \epsilon \in M$ for all real $\epsilon > 0$.

Constructibly Borel sets

- The open sets

$$U_A(a) := \{\alpha \in X(A) \mid \hat{a}_A(\alpha) > 0\},\ a \in A$$

  form a basis for the topology on $X(A)$
- If A is generated as an $\mathbb{R}$-algebra by x_i, $i \in \Omega$, the embedding $X(A) \hookrightarrow \mathbb{R}^\Omega$ defined by $\alpha \mapsto (\alpha(x_i))_{i\in\Omega}$ identifies $X(A)$ with a subspace of $\mathbb{R}^\Omega$.
- Sets of the form

$$\{b \in \mathbb{R}^\Omega \mid \sum_{i\in I}(b_i - p_i)^2 < r\},$$

  where $r, p_i \in \mathbb{Q}$ and I is a finite subset of Ω, form a basis for the product topology on $\mathbb{R}^\Omega$.

- It follows that sets of the form

$$U_A(r - \sum_{i \in I}(x_i - p_i)^2),\ r, p_i \in \mathbb{Q},\ I \text{ a finite subset of } \Omega, \quad (1)$$

  form a basis for $X(A)$.
- A subset E of $X(A)$ is called Borel if E is an element of the σ-algebra of subsets of $X(A)$ generated by the open sets.
- A subset E of $X(A)$ is said to be constructible or semialgebraic (resp., constructibly Borel) if E is an element of the algebra (resp., σ-algebra) of subsets of $X(A)$ generated by $U_A(a)$, $a \in A$.
- Constructible $\Rightarrow$ constructibly Borel $\Rightarrow$ Borel.

Countably generated algebras

If A is generated as an $\mathbb{R}$-algebra by a countable set $\{x_i \mid i \in \Omega\}$ then every Borel set of $X(A)$ is constructibly Borel.

Proof.

Sets of the form (1) form a countable basis for the topology on $X(A)$. □

Support

- For a measure space (X, Σ, μ) and a subset Y of X, we say μ is supported by Y if $E \cap Y = \emptyset \Rightarrow \mu(E) = 0 \; \forall \, E \in \Sigma$.
- In this situation, if $\Sigma' := \{E \cap Y \mid E \in \Sigma\}$, and $\mu'(E \cap Y) := \mu(E) \; \forall \, E \in \Sigma$, then Σ' is a σ-algebra of subsets of Y, μ' is a well-defined measure on (Y, Σ'), the inclusion map $i : Y \to X$ is a measurable function, and μ is the pushforward of μ' to X.
- If (Y, Σ', μ') is a measure space, (X, Σ) is a σ-algebra, $i : Y \to X$ is any measurable function, and μ is the pushforward of μ' to (X, Σ), then for each measurable function $f : X \to \mathbb{R}$, $\int f d\mu = \int (f \circ i) d\mu'$ (change in variables theorem).

Constructibly Radon measures

- A Radon measure on $X(A)$ is a positive measure μ on the σ-algebra of Borel sets of $X(A)$ which is locally finite (every point has a neighbourhood of finite measure) and inner regular (each Borel set can be approximated from within using a compact set).
- A constructibly Radon measure on $X(A)$ is a positive measure μ on the σ-algebra of constructibly Borel sets of $X(A)$ such that for, each countably generated subalgebra A' of A, the pushforward of μ to $X(A')$ via the restriction map $\alpha \mapsto \alpha|_{A'}$ is a Radon measure on $X(A')$.

From now on we consider only Radon and constructibly Radon measures having the additional property that $\hat{a}_A$ is μ-integrable (i.e., $\int \hat{a}_A d\mu$ is well-defined and finite) for all $a \in A$.

The moment problem in this general setting

- For a linear functional $L : A \to \mathbb{R}$, we consider the set of Radon or constructibly Radon measures μ on $X(A)$ such that $L(a) = \int \hat{a}_A d\mu \; \forall \, a \in A$.The moment problem is to understand this set of measures, for a given linear functional $L : A \to \mathbb{R}$. In particular, one wants to know:(i) When is this set non-empty? (ii) In case it is non-empty, when is it a singleton set?
- A linear functional $L : A \to \mathbb{R}$ is said to be positive if $L(\sum A^2) \subseteq [0, \infty)$ and M-positive for some quadratic module M of A, if $L(M) \subseteq [0, \infty)$.

Three special algebras

Let Ω is an arbitrary index set.

- As above, $A = A_\Omega := \mathbb{R}[x_i \mid i \in \Omega]$, we further define
- $B = B_\Omega := \mathbb{R}[x_i, \frac{1}{1+x_i^2} \mid i \in \Omega]$, the localization of A at the multiplicative set generated by the $1 + x_i^2$, $i \in \Omega$, and
- $C = C_\Omega := \mathbb{R}[\frac{1}{1+x_i^2}, \frac{x_i}{1+x_i^2} \mid i \in \Omega]$, the $\mathbb{R}$-subalgebra of B generated by the elements $\frac{1}{1+x_i^2}, \frac{x_i}{1+x_i^2}$, $i \in \Omega$.

By definition, A (resp., B, resp., C) is the direct limit of the $\mathbb{R}$-algebras A_I (resp., B_I, resp., C_I), I running through all finite subsets of Ω. These algebras were studied extensively in Marshall 2003 for finite Ω.Because of this, many questions about A, B and C reduce immediately to the case where Ω is finite.

Theorem

- $\sum C^2$ *is archimedean.*
- C *is naturally identified (via* $y_i \leftrightarrow \frac{1}{1+x_i^2}$ *and* $z_i \leftrightarrow \frac{x_i}{1+x_i^2}$*) with the polynomial algebra* $\mathbb{R}[y_i, z_i \mid i \in \Omega]$ *factored by the ideal generated by the polynomials* $y_i^2 + z_i^2 - y_i = (y_i - \frac{1}{2})^2 + z_i^2 - \frac{1}{4}$, $i \in \Omega$*. Consequently,* $X(C)$ *is identified naturally with* $\mathbb{S}^\Omega$*, where* $\mathbb{S} := \{(y, z) \in \mathbb{R}^2 \mid (y - \frac{1}{2})^2 + z^2 = \frac{1}{4}\}$.
- *The restriction map* $\alpha \mapsto \alpha|_C$ *identifies* $X(B)$ *with a subspace of* $X(C)$*. In terms of coordinates, this map is given by* $\alpha = (x_i)_{i\in\Omega} \mapsto \beta = (y_i, z_i)_{i\in\Omega}$*, where* $y_i := \frac{1}{1+x_i^2}$, $z_i := \frac{x_i}{1+x_i^2}$*. In particular, the image of* $X(B)$ *is dense in* $X(C)$*.*

- Elements of $X(A)$ and $X(B)$ are naturally identified with point evaluations $f \mapsto f(\alpha)$, $\alpha \in \mathbb{R}^\Omega$.
- $X(A) = X(B) = \mathbb{R}^\Omega$, not just as sets, but also as topological spaces, giving $\mathbb{R}^\Omega$ the product topology.
- $X(C) \backslash X(B) = \cup_{i \in \Omega} \Delta_i$ where $\Delta_i := \{\beta \in X(C) \mid \beta(\frac{1}{1+x_i^2}) = 0\}$.

We show how the moment problem for A_Ω reduces to understanding the extensions of a linear functional $L : A_\Omega \to \mathbb{R}$ to a positive linear functional on B_Ω and prove that positive linear functionals $L : B_\Omega \to \mathbb{R}$ correspond bijectively to constructibly Radon measures on $\mathbb{R}^\Omega$.

The main ingredients

The following is a simple modification of the argument in Marshall 2003 for arbitrary Ω

Extendibility from A to B

Suppose $L : A \to \mathbb{R}$ is an $\mathrm{Pos}_A(Y)$-positive linear functional for some closed set $Y \subseteq \mathbb{R}^\Omega$. Then L extends to an $\mathrm{Pos}_B(Y)$-positive linear functional $L : B \to \mathbb{R}$.

Positive functionals on C; Marshall 2003

Positive linear functionals $L : B \to \mathbb{R}$ restrict to positive linear functionals on C. Since the cone of sums of squares of C is archimedean, positive linear functionals $L : C \to \mathbb{R}$ are in natural one-to-one correspondence with Radon measures μ on the compact space $X(C)$ via $L \leftrightarrow \mu$ iff $L(f) = \int \hat{f}_C d\mu \; \forall f \in C$.

Main Lemma

For each positive linear functional $L : B \to \mathbb{R}$ there exists a unique Radon measure μ on $X(C)$ such that $L(f) = \int \hat{f}_C d\mu \; \forall f \in C$. This satisfies $\mu(\Delta_i) = 0 \; \forall \, i \in \Omega$ and $L(f) = \int \tilde{f} d\mu \; \forall f \in B$.

Positive functionals on B

There is a canonical one-to-one correspondence $L \leftrightarrow \nu$ given by $L(f) = \int \hat{f}_B d\nu \; \forall f \in B$ between positive linear functionals L on B and constructibly Radon measures ν on $X(B)$.

The following result extends Marshall's to the case where Ω is infinite.

Corollary

For any linear functional $L : A_\Omega \to \mathbb{R}$*, the set of constructibly Radon measures* ν *on* $\mathbb{R}^\Omega$ *satisfying* $L(f) = \int \hat{f} d\nu \; \forall f \in A_\Omega$ *is in natural one-to-one correspondence with the set of positive linear functionals* $L' : B_\Omega \to \mathbb{R}$ *extending* L*.*

The proof of the main theorem then proceeds as follows: Given L, there exists an extension of L to a linear functional L on B_Ω such that $L(\mathrm{Pos}_{B_\Omega}(Y)) \subseteq [0, \infty)$. Denote by ν the constructibly Radon measure on $\mathbb{R}^\Omega$ corresponding to this extension. Fix a countable set S in A_Ω such that $Y = X_S$. For each $g \in S$, choose $g' \in C_\Omega$ of the form $g' = g/p_g$ for some suitably chosen element $p_g = (1 + x_{j_1}^2)^{e_1} \ldots (1 + x_{j_k}^2)^{e_k}$. Let $S' = \{g' \mid g \in S\}$. Let $Q' =$ the quadratic module of C_Ω generated by S', $Q =$ the quadratic module of B_Ω generated by S. Note that Q is also the quadratic module in B_Ω generated by S', and $Q' \subseteq Q \subseteq \mathrm{Pos}_{B_\Omega}(Y)$, so $L'(Q') \subseteq [0, \infty)$ where $L' := L|_{C_\Omega}$. By Marshall 2003 there exists a Radon measure μ on $X(C_\Omega)$ supported by $X_{Q'}$ such that $L'(f) = \int \hat{f} d\mu \; \forall f \in C_\Omega$. Uniqueness implies that μ is the Radon measure on $X(C_\Omega)$ defined in Main Lemma. One checks that ν is supported by $X_{Q'} \cap X(B_\Omega) = X_Q = X_S = Y$