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Truncated and Full Moment Problems.

An intrinsic characterization of Moment Functionals in the compact case.

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1. General setting:

A : a commutative unital $\mathbb{R}$-algebra

$\mathcal{X}(A)$: the character space of A, i.e the set of all real valued $\mathbb{R}$-algebra homomorphisms on A.

- we shall always assume that $\mathcal{X}(A)$ is nonempty.

$\hat{a}$: for $a \in A$, the Gelfand transform $\hat{a} : \mathcal{X}(A) \longrightarrow \mathbb{R}$ is defined by $\hat{a}(\alpha) := \alpha(a) \quad \forall \ \alpha \in \mathcal{X}(A)$.

- Topology on $\mathcal{X}(A)$: is the weakest topology $\tau_{\mathcal{X}(A)}$ for which $\hat{a}$ is continuous, for all $a \in A$

[$\mathcal{X}(A) \subseteq \mathbb{R}^A$ inherits the product topology.]

Example (Finite dimensional setting).

$A = \mathbb{R}[x_1, \ldots, x_n]$, $\mathcal{X}(A) \simeq \mathbb{R}^n$.

$\tau_{\mathcal{X}(A)}$: Euclidean topology.

Question. Let A be a commutative unital $\mathbb{R}$- algebra, such that $\mathcal{X}(A) \neq \emptyset$.
Given a linear functional
$L : A \longrightarrow \mathbb{R}$ (with $L(1) = 1$), does there exist a Radon measure ν on $\mathcal{X}(A)$ with

$$(*) \qquad L(a) = \int_{\mathcal{X}(A)} \hat{a}(\alpha) \, d\nu(\alpha) \qquad \forall a \in A$$

such that the support of ν is compact?

2. **Main Theorem.** Let $L : A \longrightarrow \mathbb{R}$ be linear with $L(1) = 1$ and $L(A^2) \subseteq [0, \infty)$. Then $\exists !$ representing Radon measure ν_L for L (i.e ⊛ holds) with compact support if and only if

$$(**) \qquad \sup_{n \in \mathbb{N}} \sqrt[2n]{L(a^{2n})} < \infty \qquad \forall\, a \in A.$$

Moreover, in this case

$$\operatorname{support}(\nu_L) =$$

$$\left\{ \alpha \in X(A) : |\alpha(a)| \leq \sup_{n \in \mathbb{N}} \sqrt[2n]{L(a^{2n})},\ \forall a \in A \right\}.$$

Remark 1: (**) implies Carleman's condition:

$$(***) \quad \sum_{n=1}^{\infty} \left(\sqrt[2n]{L(a^{2n})} \right)^{-1} = \infty .$$

Now the proof of our main theorem follows from a recent general version of the classical Nussbaum theorem given in:

"Projective limit techniques for the infinite dimensional moment problem",
M. Infusino, T. Kuna, P. Michalski,
IEOT vol 94 # 12 (2022)
see Theorem 3.17 there!

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3. Characterization via submultiplicative seminorms on A and archimedean quadratic modules in A:

- $Q \subseteq A$ is a quadratic module if $1 \in Q$, $Q+Q \subseteq A$, $A^2 Q \subseteq Q$. Q is archimedean if $\forall a \in A$ $\exists N \in \mathbb{N}$ such that $N \pm a \in Q$.

- A function $\rho : A \longrightarrow [0,\infty)$ is a seminorm on A if $\rho(\lambda a) = |\lambda| \rho(a)$ $\forall \lambda \in \mathbb{R}$ and $\rho(a+b) \leq \rho(a) + \rho(b)$ $\forall a, b \in A$.
If in addition $\rho(ab) \leq \rho(a)\rho(b)$ $\forall a, b \in A$, then ρ is submultiplicative seminorm.
- a linear functional $L : A \longrightarrow \mathbb{R}$ is ρ-continuous iff $\exists C > 0$ such that $|L(a)| \leq C\rho(a)$ $\forall a \in A$.
- The Gelfand spectrum of ρ is
$$sp(\rho) := \{\alpha \in \mathcal{X}(A) : \alpha \text{ is } \rho\text{-continuous}\}$$

Remark 2 If p is a submultiplicative seminorm then $Sp(p) = \{\alpha \in \mathcal{X}(A) : |\alpha(a)| \leq p(a) \ \forall a \in A\}$ is compact.

$\left[Sp(p) \subseteq \prod_{a \in A} [-p(a), p(a)] \right]$ ■

Main corollary: Let $L : A \to \mathbb{R}$ be such that $L(A^2) \geq 0$ and $L(1) = 1$. TFAE:

(i) $\exists!$ representing Radon measure ν_L for L with compact support.

(ii) $\sup_{n \in \mathbb{N}} \sqrt[2n]{L(a^{2n})} < \infty \quad \forall a \in A$

(iii) L is ρ-continuous for some submultiplicative seminorm ρ on A.

(iv) $L(Q) \geq 0$ for some archimedean quadratic module $Q \subseteq A$.

In this case: • the submultiplicative seminorm defined by

$$\rho_L(a) = \sup_{n\in\mathbb{N}} \sqrt[2n]{L(a^{2n})} \quad \forall\, a\in A$$

is the smallest w.r.t which L is continuous.

• the archimedean quadratic module generated by

$$\{\rho_L(a) \pm a \;:\; a\in A\} \subseteq A$$

is the largest such that $L(Q)\geq 0$.

• Moreover

$$\operatorname{supp}(\nu_L) = \operatorname{sp}(\rho_L). \qquad \blacksquare$$

Thank

you !