Singmorelia for Mark's 63d Birthday!

~ August 7th 2024 ~

The automorphism group of a valued field.

I. Introduction and motivation.

Let (K, v) $\quad v : K^\times \longrightarrow vK := G$ ← A tot. ord. ab. gp

$v(a+b) \geq \min(v(a), v(b))$

$v(ab) = v(a) + v(b)$

Def:

$\sigma \in v\text{-}\mathrm{Aut}\, K$ iff $\forall a, b \in K^\times$

$v(a) = v(b) \Longleftrightarrow \sigma(v(a)) = \sigma(v(b))$

$v\text{-}\mathrm{Aut}\, K \leq \mathrm{Aut}(K)$.

Conditions on (K, v): • always assume that is a value group section, i.e an embedding $i : vK \hookrightarrow (K^\times, \cdot)$

•• assume that there is a residue field section, i.e an embedding

$$\iota : Kv \hookrightarrow K.$$

→ Valuation Ring $= R_K$:

$= \{x \in K \mid v(x) \geq 0\}$

Maximal Ideal $= I_K$

$= \{x \in K,\ v(x) > 0\}$

$R_K / I_K = Kv$

 7. 8. 2024

There will be 2 more conditions conditions on (K, v).

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why i . 1 ?

Kaplansky's Theory implies

$$k[G) \hookrightarrow (K, v) \hookrightarrow (k((G)), v_{min})$$

here $k := Kv$ $\quad G := vK$

$k((G)) :=$ the field of generalised formal powerseries with coefficients in k and exponents in G

$$:= \left\{ a = \sum_{g \in G} a_g t^g \; ; \; \text{support } a \subseteq G \text{ is a well ord} \right\}$$

$$\text{support } a := \{ g \in G \mid a_g \neq 0 \}.$$

$v_{min}(a) :=$ min support of a.

$a \in K^{\times}$

$$k(G) := ff(k[G])$$

Hahn field : $k(G) \leq K \leq k((G))$

 7.8.2024

Characterize v-Aut K in terms Aut k and O-Aut G

Observation 1:

For $\sigma \in \mathrm{Aut}(K)$ is val. preserving $\overset{\text{induces}}{\Longrightarrow}$

$$\sigma_G : v(a) \longmapsto v(\sigma(a)) \quad \forall a \in K^\times$$

$$\sigma_k : av \longmapsto \sigma(a)v \quad \forall a \in R_K$$

$\sigma_G \in O$-Aut G and

$\sigma_k \in$ Aut k.

<u>Note</u>: $R_{\mathbb{K}}$ where $\mathbb{K} = k((G))$

$$R_{\mathbb{K}} := k((G \geq 0))$$

$a \in R_{\mathbb{K}}$; $c(a) :=$ constant term of a

$av = c(a)$!

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(1) The Φ-map:

$$\Phi : v\text{-Aut } K \longrightarrow \mathrm{Aut}\, k \times O\text{-Aut}\, G$$
$$\sigma \longmapsto (\sigma_k, \sigma_G)$$

Φ is a group hom.

$\ker \Phi := \mathrm{int\,Aut}(K) = \{ \sigma \in v\text{-Aut} K;$
$\forall a \in K: v(a) = v(\sigma(a))$ and $\forall a \in R_K : c(a) = c(\sigma(a)) \}$

 7.8.24

1st lifting property on K:

Φ has a right section, i.e

$$\Psi : \mathrm{Aut}\, k \times 0\text{-}\mathrm{Aut}\, G \longrightarrow v\text{-}\mathrm{Aut} K$$
$$\psi(\tau, \gamma) = \sigma$$

s.t
$$\Phi \circ \Psi = \mathrm{Id}_{\mathrm{Aut}\, k \times 0\text{-}\mathrm{Aut}\, G}.$$

Example: Canonical 1LP on $\mathbb{K}$

$(\tau, \gamma) \longmapsto \sigma$ where

$$\sigma\left(\sum a_g t^g\right) := \sum \tau(a_g)\, t^{\gamma(g)} \quad (*)$$

For any $k(G) \leq K \leq \mathbb{K}$ which is invariant under $(*)$ also has the CLP 1.

$$\mathrm{Im}\, \Psi := \mathrm{Ext}\,\mathrm{Aut}\, K$$

First Decomposition:

$$v\text{-}\mathrm{Aut}\, K = \mathrm{Int}\,\mathrm{Aut}\, K \rtimes \mathrm{Ext}\,\mathrm{Aut} K$$
$$\simeq \underbrace{\mathrm{Int}\,\mathrm{Aut}\, K}_{} \rtimes (\mathrm{Aut}\, k \times 0\text{-}\mathrm{Aut} G)$$

(2) $\mathcal{E}$-map:
$$\mathcal{E} : \mathrm{Int}\,\mathrm{Aut}\, K \longrightarrow \mathrm{Hom}(G, k^{\times})$$

$$\sigma \longmapsto \left\{ g \longmapsto c\left(\frac{\sigma(t^g)}{t^g}\right) \right\}$$

$\ker \mathcal{E} := 1\text{-}\operatorname{Aut} K =$
$\{ \sigma \in \operatorname{Int}\operatorname{Aut} K \,;\, \forall a \in K : \nu(a) = \nu(\sigma(a))$ and leading coeff of a = leading coeff of $\sigma(a)$ $\}$

2nd assumption on K:
$\mathcal{E}$ has a right section i.e
$$\rho : \operatorname{Hom}(G, k^\times) \longrightarrow \operatorname{Int}\operatorname{Aut} K$$
s.t $\mathcal{E} \circ \rho = \operatorname{Id}_{\operatorname{Hom}(G, k^\times)}$

If K satisfies 2nd LP, LP2.

Example: $\chi \in \operatorname{Hom}(G, k^\times)$ define
$\sigma_\chi \in \operatorname{Int}\operatorname{Aut} K$:
$$\sigma_\chi\left(\sum a_g t^g\right) := \sum a_g \chi(g) t^g \quad (*)$$
CLP2

$\operatorname{Im} \rho = G\text{-}\exp K$

2nd decomposition: $\operatorname{Int}\operatorname{Aut} K = 1\text{-}\operatorname{Aut} K \rtimes G\text{-}\operatorname{Exp} K$

all together
$v\text{-}\operatorname{Aut} K \simeq (1\text{-}\operatorname{Aut} K) \rtimes \operatorname{Hom}(G, k^\times) \rtimes (\operatorname{Aut} k \times \operatorname{OAut}$

(3) The $\mathcal{E}_1$-map

$$\mathcal{E}_1 : \underbrace{1\text{-}\mathrm{Aut}\,K}_{\text{comp.}} \longrightarrow \underbrace{\mathrm{Hom}(G, 1+I_K)}_{\text{ptwise mult. of char}}$$

$$\sigma \longmapsto \left\{ g \longmapsto \frac{\sigma(t^g)}{t^g} \right.$$

Warning: $\mathcal{E}_1$ is NOT a group homomorphism!

Idea: If $\mathcal{E}_1$ would be injective, we could "copy" the composition group law from $1\text{-}\mathrm{Aut}\,K$ onto its bijective copy $\mathrm{Im}\,\mathcal{E}_1 \leq \mathrm{Hom}(G, 1+I_K)$!

(4) Consider the subgroup

$$1\text{-}\mathrm{Aut}^{+}K \leq 1\text{-}\mathrm{Aut}\,K$$

$$1\text{-}\mathrm{Aut}^{+}K := \{ \sigma \in 1\text{-}\mathrm{Aut}\,K ; \ \sigma \text{ is strong additive} \}$$

σ str. add: $\sigma\left(\sum a_s t^g\right) = \sum a_s \sigma(t^g)$

Observation 2: $\varepsilon_1 \restriction 1\text{-}\mathrm{Aut}^+ K$ is injective !

Image: $\varepsilon_1(1\text{-}\mathrm{Aut}^+ K) = \mathrm{Hom}^+(G, 1+I_K)$

so then

$1\text{-}\mathrm{Aut}^+ K \simeq \mathrm{Hom}^+(G, 1+I_K)$

with endow

Schillings gp. law

$\times_S$:

$$u_1 \underset{S}{\times} u_2 \,(g) := u_1(g)\,\sigma_1(u_2(g))$$

$u_1, u_2 \in \mathrm{Hom}^+(G, 1+I_K),\ g \in G$

where $\varepsilon_1(\sigma_1) = u_1$

$\mathcal{O}\text{-}\mathrm{Aut}^+ K \simeq \mathrm{Hom}^+(G, 1+I_K) \rtimes \mathrm{Hom}(G, k^*) \rtimes (\mathrm{Aut}\, k \times \mathcal{O}\text{-}\mathrm{Aut}\, G)$!

– 7 –

7.8.24