DISTINGUISHED HAHN FIELDS

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Contents

I. Introduction

1. Let (K, v) be a valued field. The aim is to study the group v-$\mathrm{Aut}(K, v)$ of valuation preserving automorphisms of (K, v): $\sigma \in \mathrm{Aut}\,K$ is valuation preserving if $\forall a, b \in K : v(a) = v(b) \Rightarrow v(\sigma(a)) = v(\sigma(b))$.
2. We shall always assume that K admits both a residue field and a value group section: $Kv \overset{i}{\hookrightarrow} K$ and $G := v(K^\times) \hookrightarrow (K^\times, \cdot)$. Set $k := i(Kv)$.

- Under these (and other) conditions, a theorem of Kaplansky gives

$$(K, v) \overset{\iota}{\hookrightarrow} (k((G)), v_{\min})$$

such that $k(G) \subseteq \iota(K) \subseteq k((G))$ where $k(G) := \mathrm{ff}(k[G])$ is the *minimal Hahn field*.

3. (a) So we study v-$\mathrm{Aut}\,K$ for a Hahn field $k(G) \subseteq K \subseteq k((G))$ and, as we shall see, the best decomposition result for v-$\mathrm{Aut}\,K$ is under further assumptions on K, the lifting properties, which are the topic of this talk. Let us call a Hahn field satisfying those lifting properties a *distinguished Hahn field*.
   (b) To achieve our aim we will then, in the spirit of Mourgues and Ressayre, study necessary and sufficient conditions on a valued field (K, v) (with $Kv = k$ and $v(K) \simeq G$) to embed in $k((G))$ as a distinguished Hahn field.
   (c) Finally, a further project would be to understand how v-$\mathrm{Aut}\,K$ varies with v (in particular coarsenings, independent valuations, henselian valuations etc.). In particular v-$\mathrm{Aut}_k\,K \leq \mathrm{Gal}(K/k)$.

II. THE FIRST LIFTING PROPERTY

1. In [KMP17] we **characterise valuation preserving automorphisms** by the conditions

$$\begin{cases} \text{the map } \sigma_G\colon v(a) \mapsto v(\sigma(a)) \text{ for } a \in K \text{ is in } o\text{-Aut}\, G; \\ \text{the map } \sigma_v\colon av \mapsto \sigma(a)v \text{ for } a \in R_K \text{ is in } \operatorname{Aut} k. \end{cases}$$

   where we denote by R_K the valuation ring of K and by I_K the valuation ideal.
2. **Identification of Kv with k.** Let c denote the canonical identification: for $a \in k\left(\!\left(G^{\geq 0}\right)\!\right)$ let $c(a)$ be the constant term of a and set $f_c\colon Kv \to k$ the corresponding isomorphism. Note that other isomorphisms $f\colon Kv \to k$ are of the form $f_\tau = \tau \circ f_c$ for $\tau \in \operatorname{Aut} k$. So, without loss of generality, we shall work with f_c and call it the *canonical identification of Kv with k*.
3. **The Φ-map.** The map

$$\begin{array}{rccc} \Phi = \Phi_c\colon & v\text{-Aut}\, K & \to & \operatorname{Aut} k \times o\text{-Aut}\, G \\ & \sigma & \mapsto & (\sigma_k, \sigma_G) \end{array}$$

   is a group homomorphism with

$$\begin{aligned} \ker \Phi &=: \operatorname{Int}\operatorname{Aut} K \\ &= \{\sigma \in \operatorname{Aut} K \mid \forall a \in K : v(a) = v(\sigma(a)) \text{ and } \forall a \in R_K : c(a) = c(\sigma(a))\}. \end{aligned}$$

Definition II.1. K has the *first lifting property (1LP)* if Φ admits a section, i.e., a group homomorphism

$$\Psi\colon \operatorname{Aut} k \times o\text{-Aut}\, G \to v\text{-Aut}\, K$$

such that $\Phi \circ \Psi = \mathrm{id}_{\operatorname{Aut} k \times o\text{-Aut}\, G}$.

Example II.1. $k(\!(G)\!)$ admits the *canonical 1LP* . Indeed

$$\begin{array}{rccc} \Psi_c\colon & \operatorname{Aut} k \times o\text{-Aut}\, G & \to & v\text{-Aut}\, k(\!(G)\!) \\ & (\tau, \gamma) & \mapsto & \sigma\colon \sigma\left(\sum a_g t^g\right) = \sum \tau(a_g) t^{\gamma(g)} \quad (*) \end{array}$$

is a section of Φ.

Definition II.2. A Hahn field *admits the canonical 1LP* if it is stable under $(*)$.

Example II.2.
- Let $K = k(\!(\mathcal{F})\!)$ be the k-Hull of a field family. Then K admits the canonical 1LP if and only if $\mathcal{F}$ is stable under $o\text{-Aut}\, G$.
- $k(G)$ is not a k-Hull, so not a Rayner field ($\exp(t^g) \notin k(G)$, for $\operatorname{char} k = 0$) but has the canonical 1LP.

Example II.3. Let $\mathcal{F}_0 = \{\text{finite subsets of } G\} \cup \{S\}$ where $G = \coprod_{-\mathbb{N}} \mathbb{Q}$ and $S = \{\mathbb{1}_{-n} : n \in \mathbb{N}\}$. Let $\mathcal{F} = \overline{\mathcal{F}_0}$ be the Rayner closure of $\mathcal{F}_0$. Then $\mathcal{F}$ is not stable under $\gamma \in o\text{-Aut}\, G$ defined by $\gamma(g) = \frac{1}{2} g$ and so $k(\!(\mathcal{F})\!)$ does not have the canonical 1LP. $\square$

Question. Does $k((\mathcal{F}))$ from Example II.3 admit other 1LPs? That is, is there still a (different) section of Φ_c? We understand that if Ψ is a right section, then all conjugates of Ψ by elements of $\operatorname{Int}\operatorname{Aut} K$ are also right sections. Yet this does not answer the questions, as there might still be sections in different conjugacy classes.

4. **Definition II.3.** $\operatorname{im}\Psi =: \operatorname{Ext}_\Psi \operatorname{Aut} K$.
5. **Facts**

   $\operatorname{Int}\operatorname{Aut} K \trianglelefteq v\text{-}\operatorname{Aut} K$, $\operatorname{Ext}_\Psi \operatorname{Aut} K \simeq \operatorname{Aut} k \times o\text{-}\operatorname{Aut} G$ and $v\text{-}\operatorname{Aut} K \simeq \operatorname{Int}\operatorname{Aut} K \rtimes \operatorname{Ext}_\Psi \operatorname{Aut} K$. So we have the **first decomposition Theorem:**

   $$v\text{-}\operatorname{Aut} K \simeq \operatorname{Int}\operatorname{Aut} K \rtimes (\operatorname{Aut} k \times o\text{-}\operatorname{Aut} G) \tag{II.1}$$

   The group o-$\operatorname{Aut} G$ is subject to a total analysis in another paper.
6. Our next task is to analyse $\operatorname{Int}\operatorname{Aut} K$.

III. The second lifting property

Consider the map

$$\begin{array}{rccc} X := X_c\colon & \operatorname{Int}\operatorname{Aut} K & \to & \operatorname{Hom}(G, k^\times)^{1} \\ & \sigma & \mapsto & x_\sigma\colon g \mapsto c(t^{-g}\sigma(t^g)) \end{array}$$

Then X is a well defined group homomorphism. We define:

$$\begin{aligned} 1\text{-}\operatorname{Aut} K &:= \ker X \\ &= \left\{\sigma \;\middle|\; \forall a \in K: \quad v(a) = v(\sigma(a)),\ a_{v(a)} = \sigma(a)_{v(a)} \right\} \end{aligned}$$

Definition III.1. K has the *second lifting property* if X admits a right section, i.e., a group homomorphism $P : \operatorname{Hom}(G, k^\times) \to \operatorname{Int}\operatorname{Aut} K$ such that $X \circ P = \operatorname{id}_{\operatorname{Hom}(G,k^\times)}$.

Example III.1. $k((G))$ has the canonical second lifting property: for $x \in \operatorname{Hom}(G, k^\times)$ define $\sigma_x \in v\text{-}\operatorname{Aut} k((G))$ by $\sigma_x\left(\sum a_g t^g\right) = \sum a_g x(g) t^g$ $(**)$. Then $\sigma_x \in \operatorname{Int}\operatorname{Aut} k((G))$.

Definition III.2. K has the canonical 2LP if it is stable under $(**)$.

Example III.2. All k-hulls have the canonical 2LP. And so does $k(G)$.

Definition III.3. Define $G_P\text{-}\operatorname{Exp} K := \operatorname{im} P$.

Facts: $1\text{-}\operatorname{Aut} K \trianglelefteq \operatorname{Int}\operatorname{Aut} K$; $G_P\text{-}\operatorname{Exp} K \simeq \operatorname{Hom}(G, k^\times)$ and $\operatorname{Int}\operatorname{Aut} K = 1\text{-}\operatorname{Aut} K \rtimes G_P\text{-}\operatorname{Exp} K$ which yield the **second decomposition Theorem**:

$$\operatorname{Int}\operatorname{Aut} K \simeq 1\text{-}\operatorname{Aut} K \rtimes \operatorname{Hom}(G, k^\times). \tag{III.1}$$

All together, for a distinguishes Hahn field K we have

$$v\text{-}\operatorname{Aut} K \simeq (1\text{-}\operatorname{Aut} K \rtimes \operatorname{Hom}(G, k^\times)) \rtimes (\operatorname{Aut} k \times o\text{-}\operatorname{Aut} G). \tag{III.2}$$

All in terms of the valuation invariants k and G except for the first factor $1\text{-}\operatorname{Aut} K$ which we now finally attack in the last part of this talk.

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[1] Seen as an abelian group under pointwise multiplication of characters.

IV. 1-Aut K

1. We would want to analyse 1-Aut K with the help of an appropriate map, as we did for v-Aut K and Int Aut K.
2. There is indeed a map:

$$\begin{array}{rccc} X_1\colon & \text{1-Aut}\, K & \to & \mathrm{Hom}(G, 1+I_K)^{2} \\ & \sigma & \mapsto & (g \mapsto t^{-g}\sigma(t^g)) \end{array}$$

   However, X_1 is not a group homomorphism.
3. The idea is now: if X_1 would be injective, we could "copy" the group operation from 1-Aut K onto its isomorphic copy $\operatorname{im} X_1 \subseteq \mathrm{Hom}(G, 1 + I_K)$.
4. There is a privileged subgroup $\text{1-Aut}^+ K \leq \text{1-Aut}\, K$ of strongly additive 1-automorphisms for which $X_1|_{\text{1-Aut}^+ K}$ is indeed injective [since a strongly additive map is completely determined by its value on the terms, in particular, $\sigma \in \text{1-Aut}^+ K$ is determined by $\sigma(t^g)$ for $g \in G$].
5. Define therefore $\mathrm{Hom}^+(G, 1 + I_K) := X_1(\text{1-Aut}^+ K)$.

Definition IV.1. (i) $x \in \mathrm{Hom}(G, 1 + I_K)$ is K*-summable* if, for all $a = \sum a_g t^g \in K$ we have $\sum a_g x(g) t^g \in K$ $(***)$.

(ii) $\mathrm{Hom}^+(G, 1+I_K)$ is endowed with the multiplication (defined by Schilling [Sch44] for the case of $K = \mathbb{L} = k(\!(\mathbb{Z})\!)$)

$$(u_1 \times_S u_2)(g) = u_1(g) \cdot \sigma_1(u_2(g))$$

for all $g \in G$, $u_1, u_2 \in \mathrm{Hom}^+(G, 1 + I_K)$ and σ_1 is determined by $X_1(\sigma_1) = u_1$.

All together we get the following decomposition in terms of the valuation invariants

$$v\text{-Aut}^+ K \simeq ((\mathrm{Hom}^+(G, 1 + I_K), \times_S) \rtimes \mathrm{Hom}(G, k^\times)) \rtimes (\mathrm{Aut}\, k \times o\text{-Aut}\, G). \tag{IV.1}$$

References

[KMP17] S. Kuhlmann, M. Matusinski, and F. Point. "The valuation difference rank of a quasi-ordered difference field". In: *Groups, modules, and model theory – surveys and recent developments* (2017), pp. 399–414.

[Sch44] O. F. G. Schilling. "Automorphisms of fields of formal power series". In: *Bull. Amer. Math. Soc.* 50.12 (1944), pp. 892–901.

[2] Again with pointwise multiplication of characters.