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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 9

These exercises will be collected Tuesday 22 December in the mailbox number 15 of the Mathematics department.

Definition 0.1 *A* ***first order formula*** *in the language of real closed fields is obtained as follows recursively:*

1. *if* $f(\underline{x}) = \mathbb{Q}[x_1, \ldots, x_n]$, $n > 1$, *then* $f(\underline{x}) \geq 0$, $f(\underline{x}) > 0$, $f(\underline{x}) = 0$, $f(\underline{x}) \neq 0$ *are first order formulas (with free variables* $\underline{x} = (x_1,...,x_n)$*);*
2. *if* Φ *and* Ψ *are first order formulas, then* $\Phi \vee \Psi$, $\Phi \wedge \Psi$ *and* $\neg\Phi$ *are also first order formulas (with free variables given by the union of the free variables of* Φ *and the free variables of* Ψ*);*
3. *if* Φ *is a first order formula then*
$$\exists x\, \Phi \ \textit{ and } \ \forall x\, \Phi$$
are first order formulas (with same free variables as Φ *minus* $\{x\}$*).*

The formulas obtained using just 1. and 2. are called ***quantifier free****.*

Let R *be a real closed field,* $n \geq 1$. *A subset* $A \subset R^n$ *is said to be* ***definable*** *(with parameters from* R*) in* R *if there is a first order formula* $\Phi(\underline{t},\underline{x})$ *with parameters* $\underline{t} \in R^m$ *and free variables* $\underline{x} = (x_1,...,x_n)$*, such that*
$$A = \{\underline{x} \in R^n : \Phi(\underline{t},\underline{x}) \text{ is true in } R\}.$$

Proposition 0.2 *For any real closed field* R*, the class of definable sets (with parameters) in* R *coincides with the class of semialgebraic sets.*

Theorem 0.3 (Tarski's quantifier elimination theorem for real closed fields) *Every first order formula in the language of real closed fields is equivalent to a quantifier free formula.*

Theorem 0.4 (Tarski-Seidenberg, geometric version) *Consider the projection map*
$$\begin{array}{cccc} \Pi : & R^{m+n} & \rightarrow & R^m \\ & (\underline{x},\underline{y}) & \mapsto & \underline{x}. \end{array}$$
Then, for any semi-algebraic set $A \subset R^{m+n}$, $\Pi(A)$ *is a semi-algebraic subset of* R^m.

1. We suppose R endowed with the interval topology, which coincides with the euclidean topology (see Ü.B. 5).
   (a) Show that the closure $\mathrm{Cl}(A)$, the interior $\mathrm{Int}(A)$ and the boundary ∂A of a semi-algebraic set A ($\subset R^n$ for some n), are semi-algebraic:
   (i) by giving a first order formula defining them;
   (*Hint: use the formal definition of a point belonging to a closed set*;)
   (ii) by a geometrical description.

   (b) Describe the closure $\mathrm{Cl}(A)$ of the following semi-algebraic set
   $$A = \{(x,y) \in R^2 \mid x^3 - x^2 - y^2 > 0\}.$$
   Note that the closure of a semi-algebraic set is not obtained just by relaxing the strict inequalities.

2. Let A, B be semi-algebraic sets. Show that:
   (a) any polynomial map $f : A \to B$ is semi-algebraic;

   (b) any regular rational map (i.e. a map whose coordinates are rational functions with denominators that vanish nowhere in A) $f : A \to B$ is semi-algebraic;

   (c) if $f : A \to R$ and $g : A \to R$ are semi-algebraic maps, then so are the maps $\max(f,g)$, $\min(f,g)$ and $|f|$;

   (d) if $f : A \to R$ is semi-algebraic and $f \geq 0$, then so is $\sqrt{f}$.

3. Prove the following theorem:
   Theorem 0.5 *Let R be a real closed field, and A, B, C and D be some nonempty semi-algebraic sets. Then:*
   (a) for any semi-algebraic maps $f : A \to B$ and $g : B \to C$, the map $g \circ f : A \to C$ is also semi-algebraic;

   (b) for any semi-algebraic maps $f : A \to B$ and $g : C \to D$, the map
   $$\begin{array}{rccc} f \times g : & A \times C & \to & B \times D \\ & (a,c) & \mapsto & (f(a),g(c)) \end{array}$$
   is also semi-algebraic;

   (c) Let $f : A \to B$ be a semi-algebraic map.
   (i) For any semi-algebraic subset $S \subset A$, its image $f(S)$ is also semi-algebraic.
   (ii) For any semi-algebraic subset $T \subset B$, its preimage $f^{-1}(T)$ is also semi-algebraic.

   (d) Define $\mathcal{S}(A) := \{f : A \to R \mid f$ is a semi-algebraic map$\}$. Then $\mathcal{S}(A)$ endowed with pointwise addition and pointwise multiplication is a commutative ring.