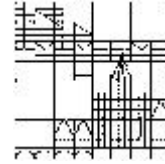


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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 9

These exercises will be collected Tuesday 22 December in the mailbox number 15 of the Mathematics department.

Definition 0.1 A *first order formula* in the language of real closed fields is obtained as follows recursively:

1. if $f(\underline{x}) = \mathbb{Q}[x_1, \dots, x_n]$, $n > 1$, then $f(\underline{x}) \geq 0$, $f(\underline{x}) > 0$, $f(\underline{x}) = 0$, $f(\underline{x}) \neq 0$ are first order formulas (with free variables $\underline{x} = (x_1, \dots, x_n)$);
2. if Φ and Ψ are first order formulas, then $\Phi \vee \Psi$, $\Phi \wedge \Psi$ and $\neg\Phi$ are also first order formulas (with free variables given by the union of the free variables of Φ and the free variables of Ψ);
3. if Φ is a first order formula then

$$\exists x \Phi \text{ and } \forall x \Phi$$

are first order formulas (with same free variables as Φ minus $\{x\}$).

The formulas obtained using just 1. and 2. are called **quantifier free**.

Let R be a real closed field, $n \geq 1$. A subset $A \subset R^n$ is said to be **definable** (with parameters from R) in R if there is a first order formula $\Phi(\underline{t}, \underline{x})$ with parameters $\underline{t} \in R^m$ and free variables $\underline{x} = (x_1, \dots, x_n)$, such that

$$A = \{\underline{x} \in R^n : \Phi(\underline{t}, \underline{x}) \text{ is true in } R\}.$$

Proposition 0.2 For any real closed field R , the class of definable sets (with parameters) in R coincides with the class of semialgebraic sets.

Theorem 0.3 (Tarski's quantifier elimination theorem for real closed fields) Every first order formula in the language of real closed fields is equivalent to a quantifier free formula.

Theorem 0.4 (Tarski-Seidenberg, geometric version) Consider the projection map

$$\begin{aligned} \Pi : R^{m+n} &\rightarrow R^m \\ (\underline{x}, \underline{y}) &\mapsto \underline{x}. \end{aligned}$$

Then, for any semi-algebraic set $A \subset R^{m+n}$, $\Pi(A)$ is a semi-algebraic subset of R^m .

1. We suppose R endowed with the interval topology, which coincides with the euclidean topology (see Ü.B. 5).
 - (a) Show that the closure $\text{Cl}(A)$, the interior $\text{Int}(A)$ and the boundary ∂A of a semi-algebraic set $A \subset R^n$ for some n , are semi-algebraic:
 - (i) by giving a first order formula defining them;
(Hint: use the formal definition of a point belonging to a closed set);
 - (ii) by a geometrical description.

(b) Describe the closure $\text{Cl}(A)$ of the following semi-algebraic set

$$A = \{(x,y) \in R^2 \mid x^3 - x^2 - y^2 > 0\}.$$

Note that the closure of a semi-algebraic set is not obtained just by relaxing the strict inequalities.

2. Let A, B be semi-algebraic sets. Show that:
 - (a) any polynomial map $f : A \rightarrow B$ is semi-algebraic;
 - (b) any regular rational map (i.e. a map whose coordinates are rational functions with denominators that vanish nowhere in A) $f : A \rightarrow B$ is semi-algebraic;
 - (c) if $f : A \rightarrow R$ and $g : A \rightarrow R$ are semi-algebraic maps, then so are the maps $\max(f,g)$, $\min(f,g)$ and $|f|$;
 - (d) if $f : A \rightarrow R$ is semi-algebraic and $f \geq 0$, then so is \sqrt{f} .

3. Prove the following theorem:

Theorem 0.5 *Let R be a real closed field, and A, B, C and D be some nonempty semi-algebraic sets. Then:*

(a) *for any semi-algebraic maps $f : A \rightarrow B$ and $g : B \rightarrow C$, the map $g \circ f : A \rightarrow C$ is also semi-algebraic;*

(b) *for any semi-algebraic maps $f : A \rightarrow B$ and $g : C \rightarrow D$, the map*

$$\begin{aligned} f \times g : A \times C &\rightarrow B \times D \\ (a,c) &\mapsto (f(a),g(c)) \end{aligned}$$

is also semi-algebraic;

(c) *Let $f : A \rightarrow B$ be a semi-algebraic map.*

- (i) *For any semi-algebraic subset $S \subset A$, its image $f(S)$ is also semi-algebraic.*
- (ii) *For any semi-algebraic subset $T \subset B$, its preimage $f^{-1}(T)$ is also semi-algebraic.*

(d) *Define $\mathcal{S}(A) := \{f : A \rightarrow R \mid f \text{ is a semi-algebraic map}\}$. Then $\mathcal{S}(A)$ endowed with pointwise addition and pointwise multiplication is a commutative ring.*