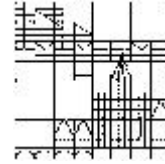


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## Übungen zur Vorlesung Reelle algebraische Geometrie

### Blatt 2

*These exercises will be collected Tuesday 3 November either in one of the mailboxes of the Mathematics department, or during the break of the lecture.*

1. **Definition 0.1** An ordered field  $(K, \leq)$  is:

- (i) **Dedekind complete (vollständig)**, if for any pair of non empty subsets  $L$  and  $U$  of  $K$  such that  $L \leq U$  (i.e.  $\lambda \leq \mu$  for any  $\lambda \in L$  and any  $\mu \in U$ ), there exists  $\alpha \in K$  such that  $L \leq \alpha \leq U$ ;
- (ii) **archimedean** if for any  $\alpha \in K$ , there exists  $n \in \mathbb{N}$  such that  $\alpha \leq n$ ;
- (iii) **complete** if any Cauchy sequence converges.

(a) Let  $(K, \leq)$  be an ordered field. Show that  $K$  is Dedekind complete if and only if it is archimedean and complete.

(b) Let  $(K, \leq)$  be an archimedean ordered field. Show that  $\mathbb{Q}$  is **dense** in  $(K, \leq)$ , i.e.  $\forall \alpha < \beta \in K, \exists r \in \mathbb{Q}, \alpha < r < \beta$ .

(c) Let  $(K, \leq)$  be an archimedean ordered field. Let  $\rho : K \rightarrow \mathbb{R}$  be the map which to each element  $\alpha \in K$  associates the uniquely determined real number  $\rho(\alpha) \in \mathbb{R}$  such that  $U_\alpha \leq \rho(\alpha) \leq O_\alpha$ , where:

$$U_\alpha := \{r \in \mathbb{Q} \mid r < \alpha\} \text{ and } O_\alpha := \{r \in \mathbb{Q} \mid \alpha \leq r\}.$$

Show that:

- (i)  $\rho$  is a ring homomorphism, and therefore a field embedding;
- (ii) for any  $\alpha, \beta \in K$ ,  $\alpha \leq \beta$  if and only if  $\rho(\alpha) \leq \rho(\beta)$ . Therefore  $\rho$  preserves the ordering.

This completes the proof of **Hölder's theorem**.

(d) Let  $(K, \leq)$  be a Dedekind complete ordered field. Deduce that  $K$  is isomorphic to  $\mathbb{R}$  as ordered field (Hint: recall that  $\mathbb{R}$  is complete).

Therefore,  $(\mathbb{R}, \leq)$  is the unique Dedekind complete ordered field up to isomorphism.

2. **Definition 0.2** A **cone** of a field  $K$  is a subset  $P$  of  $K$  such that:

(i)  $\alpha, \beta \in P \Rightarrow \alpha + \beta \in P$ ;

(ii)  $\alpha, \beta \in P \Rightarrow \alpha \cdot \beta \in P$ ;

(iii)  $\alpha \in K \Rightarrow \alpha^2 \in P$ .

The cone is said to be **proper** if in addition:

(iv)  $-1 \notin P$ .

(a) Under conditions (i), (ii) and (iii), show that  $P$  is proper if and only if:

$$(iv)' \quad P \cap -P = \{0\}.$$

(b) Given an ordered field  $(K, \leq)$ , consider its subset  $P = \{\alpha \in K \mid \alpha \geq 0\}$  of non negative elements. Show that this a proper cone satisfying:

(v)  $P \cup -P = K$  (where  $-P := \{\alpha \in K \mid -\alpha \in P\}$ ).

The set  $P = \{\alpha \in K \mid \alpha \geq 0\}$  is called the **positive cone** of  $K$ .

(c) Show that, conversely, if  $P$  is a proper cone of a field  $K$  satisfying (v), then  $K$  is ordered by

$$\alpha \leq \beta \Leftrightarrow \beta - \alpha \in P.$$

(d) Deduce that there is a bijective correspondance between orderings of  $K$  and positives cones of  $K$ .

3. **Notation 0.3** The **set of sums of squares** of elements of a field  $K$  is denoted by  $\sum K^2$ .

(a) Show that the set  $\sum K^2$  is a cone, and is contained in any cone of  $F$ .

(b) A field  $K$  is said to be **real** if it admits at least one order. Show that, if  $K$  is real, then

$$-1 \notin \sum K^2.$$

(c) Show that if a field  $K$  is algebraically closed, then it is not real.

(d) Show that if  $(K, P)$  is an ordered field,  $F$  another field,  $\varphi : F \rightarrow K$  a field homomorphism, then  $Q := \varphi^{-1}(P)$  is an ordering of  $F$ . In this case, we say that  $P$  is an **extension** of  $Q$  (where  $Q$  is the **pullback** of  $P$ ).

(e) Show that if  $P$  and  $Q$  are orderings of a field  $K$  with  $P \subset Q$  then  $P = Q$ .

(f) In particular, show that if  $P = \sum K^2$  happens to be a positive cone of  $K$ , then it is the only ordering of  $K$ .

(g) As examples, consider the fields  $\mathbb{R}$  and  $\mathbb{Q}$  and show that they admit a unique ordering.