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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 2

These exercises will be collected Tuesday 3 November either in one of the mailboxes of the Mathematics department, or during the break of the lecture.

1. **Definition 0.1** *An ordered field $(K, \leq)$ is:*
(i) ***Dedekind complete (vollständig)****, if for any pair of non empty subsets L and U of K such that $L \leq U$ (i.e. $\lambda \leq \mu$ for any $\lambda \in L$ and any $\mu \in U$), there exists $\alpha \in K$ such that $L \leq \alpha \leq U$;*
(ii) ***archimedean*** *if for any $\alpha \in K$, there exists $n \in \mathbb{N}$ such that $\alpha \leq n$;*
(iii) ***complete*** *if any Cauchy sequence converges.*

   (a) Let $(K, \leq)$ be an ordered field. Show that K is Dedekind complete if and only if it is archimedean and complete.

   (b) Let $(K, \leq)$ be an archimedean ordered field. Show that $\mathbb{Q}$ is **dense** in $(K, \leq)$, i.e. $\forall \alpha < \beta \in K, \exists r \in \mathbb{Q}, \alpha < r < \beta$.

   (c) Let $(K, \leq)$ be an archimedean ordered field. Let $\rho : K \to \mathbb{R}$ be the map which to each element $\alpha \in K$ associates the uniquely determined real number $\rho(\alpha) \in \mathbb{R}$ such that $U_\alpha \leq \rho(\alpha) \leq O_\alpha$, where:
$$U_\alpha := \{r \in \mathbb{Q} \mid r < \alpha\} \text{ and } O_\alpha := \{r \in \mathbb{Q} \mid \alpha \leq r\}.$$
   Show that:
(i) ρ is a ring homomorphism, and therefore a field embedding;
(ii) for any $\alpha, \beta \in K$, $\alpha \leq \beta$ if and only if $\rho(\alpha) \leq \rho(\beta)$. Therefore ρ preserves the ordering.
This completes the proof of **Hölder's theorem**.

   (d) Let $(K, \leq)$ be a Dedekind complete ordered field. Deduce that K is isomorphic to $\mathbb{R}$ as ordered field (Hint: recall that $\mathbb{R}$ is complete).

   Therefore, $(\mathbb{R}, \leq)$ *is the unique Dedekind complete ordered field up to isomorphism.*

2. **Definition 0.2** *A **cone** of a field K is a subset P of K such that:*
(i) $\alpha,\beta \in P \Rightarrow \alpha+\beta \in P$*;*
(ii) $\alpha,\beta \in P \Rightarrow \alpha.\beta \in P$*;*
(iii) $\alpha \in K \Rightarrow \alpha^2 \in P$*.*
*The cone is said to be **proper** if in addition:*
(iv) $-1 \notin P$*.*

(a) Under conditions (i), (ii) and (iii), show that P is proper if and only if:

$$\text{(iv)' } P \cap -P = \{0\}.$$

(b) Given an ordered field $(K, \leq)$, consider its subset $P = \{\alpha \in K \mid \alpha \geq 0\}$ of non negative elements. Show that this a proper cone satisfying:
(v) $P \cup -P = K$ (where $-P := \{\alpha \in K \mid -\alpha \in P\}$).

The set $P = \{\alpha \in K \mid \alpha \geq 0\}$ is called the **positive cone** of K.

(c) Show that, conversely, if P is a proper cone of a field K satisfying (v), then K is ordered by

$$\alpha \leq \beta \Leftrightarrow \beta - \alpha \in P.$$

(d) Deduce that there is a bijective correspondance between orderings of K and positives cones of K.

3. **Notation 0.3** *The **set of sums of squares** of elements of a field K is denoted by* $\sum K^2$.
(a) Show that the set $\sum K^2$ is a cone, and is contained in any cone of F.

(b) A field K is said to be **real** if it admits at least one order. Show that, if K is real, then

$$-1 \notin \sum K^2.$$

(c) Show that if a field K is algebraically closed, then it is not real.

(d) Show that if (K,P) is an ordered field, F another field, $\varphi : F \to K$ a field homomorphism, then $Q := \varphi^{-1}(P)$ is an ordering of F. In this case, we say that P is an **extension** of Q (where Q is the **pullback** of P).

(e) Show that if P and Q are orderings of a field K with $P \subset Q$ then $P = Q$.

(f) In particular, show that if $P = \sum K^2$ happens to be a positive cone of K, then it is the only ordering of K.

(g) As examples, consider the fields $\mathbb{R}$ and $\mathbb{Q}$ and show that they admit a unique ordering.