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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 15

1. **Definition 0.1** *Let* $(G, +, \leq)$ *be an ordered abelian group. For any* $x \in G$, $x \neq 0$, *we define*

$$C_x := \bigcap\{C \text{ convex subgroup of } G,\ x \in C\}.$$

This is the smallest convex subgroup of G which contains x.
We also denote

$$D_x := \bigcup\{C \text{ convex subgroup of } G,\ x \notin C\}.$$

Proposition 0.2 *(a)* D_x *is the biggest convex subgroup of G which does not contain x.*
(b) The extension from D_x *to* C_x *is a* ***jump*** *(=* ***Sprung****), i.e. for any* $D_x \subseteq C \subseteq C_x$ *with C convex, then* $C = D_x$ *or* $C = C_x$. *We write* $D_x \prec C_x$.
(c) Consequently, the ordered abelian group $B_x := C_x/D_x$ *has no proper non trivial convex subgroup.*

Proof.
(a) D_x is non empty since it contains $\{0\}$. Consider $a,b \in D_x$, there exist convex subgroups A and B of G which do not contain x and such that $a \in A$ and $b \in B$. Since convex subgroups are totally ordered by inclusion (see ÜA Blatt 14), we have either $A \subseteq B$ or $B \subseteq A$. Suppose for instance that $B \subseteq A$. Then $b \in A$, and so $a + b \in A \subseteq D_x$. Thus D_x is a subgroup of G. Moreover, since $x \in C_x$ but $x \notin D_x$, we have $D_x \subsetneq C_x$. Therefore D_x is a proper convex subgroup of C_x. Moreover, D_x is the biggest convex subgroup which do not contain x (if not, this would contradict the fact that D_x is the union of all the convex subgroups which do not contain x).

(b) Consider a convex subgroup C of G such that $D_x \subseteq C \subseteq C_x$. Then there are two cases. Either $x \in C$, which implies that $C_x \subseteq C$ and so $C_x = C$. Or $x \notin C$, which implies that $C \subseteq D_x$ and so $C = D_x$.

(c) The existence of a proper non trivial subgroup of B_x would imply the existence of a convex subgroup C of C_x such that $D_x \subsetneq C \subsetneq C_x$, which is impossible by the preceding question.

2. **Definition 0.3** *An ordered abelian group $(A, +, \leq)$ is said to be **archimedian** if for any $a_1,a_2 \in A$ with $a_1 \neq 0$ and $a_2 \neq 0$, there exists $n \in \mathbb{N}$ such that $n|a_1| \geq |a_2|$ and $n|a_2| \geq |a_1|$ (where $|a| := \max\{a; -a\}$).*

   Proposition 0.4 *An ordered abelian group $(A, +, \leq)$ is archimedian if and only if it has no non trivial proper convex subgroup.*

   Proof.
   Consider an ordered abelian group $(A, +, \leq)$ and suppose that it has a proper non trivial convex subgroup $\{0\} \subsetneq C \subsetneq A$. Then take $x \in C$ and $y \in G \setminus C$: x and y would not be archimedean equivalents. Indeed, if they were equivalents, we would have some $n \in \mathbb{N}$ such that $n|x| \geq |y|$ which would imply that $y \in C$ by convexity.
   Conversely, suppose that A is non archimedean. This means that there exist x and y which are not archimedean equivalents, i.e. such that for instance $x <<^+ y$. Then we claim that the corresponding convex subgroups C_x, D_y and C_y are such that $\{0\} \subsetneq C_x \subseteq D_y \subsetneq C_y$. Indeed, it suffices to notice that the set $\{z \in G \mid \exists n \in \mathbb{N}, n|z| \geq |x|\}$ is a convex subgroup of G which contains x without containing y.
3. **Definition 0.5** • *Given an ordered abelian group $(G, +, \leq)$, two nonzero elements $x,y \in G$ are said to be **archimedian equivalent**, denoted by $x \sim^+ y$, if there exists $n \in \mathbb{N}$ such that $n|x| \geq |y|$ and $n|y| \geq |x|$.*
   • *Otherwise, given two nonzero elements $x,y \in G$, if we have $n|x| < |y|$ for any $n \in \mathbb{N}$, then we denote $x <<^+ y$.*

   Proposition 0.6 *The relation $\sim^+$ is compatible with the relation $<<^+$ in the following sense: for any nonzero $x,y,z \in G$,*

   if $x <<^+ y$ and $z \sim^+ x$, then $z <<^+ y$;
   if $x <<^+ y$ and $z \sim^+ y$, then $x <<^+ z$.

   Proof.
   Consider $x,y,z \in G$ such that $x <<^+ y$ and $z \sim^+ x$. This means that for all $m \in \mathbb{N}$, $m|x| < |y|$, and that there exists $n \in \mathbb{N}$ such that $n|x| \geq |z|$. So for any $k \in \mathbb{N}$, $k|z| \leq kn|x| < |y|$. Thus $z <<^+ y$.
   Consider now $x,y,z \in G$ such that $x <<^+ y$ and $z \sim^+ y$. This means that for all $m \in \mathbb{N}$, $m|x| < |y|$, and that there exists $n \in \mathbb{N}$ such that $n|z| \geq |y|$. So for any $k \in \mathbb{N}$, $kn|x| < |y| \leq n|z|$, which implies that $k|x| < |z|$. Thus $x <<^+ z$.

4. Given an ordered abelian group $(G, +, \leq)$, we consider the set $\Gamma := G \setminus \{0\}/\sim^+$ of its archimedean equivalence classes. We define a relation on Γ by, for any nonzero $x,y \in G$,

$$[y] <_\Gamma [x] \Leftrightarrow x <<^+ y.$$

Proposition 0.7 *(a) The relation $\leq_\Gamma$ is a total ordering on Γ.*
The ordered set $(\Gamma := G \setminus \{0\}/\sim^+, \leq_\Gamma)$ is called the ***rank*** *of G, denoted by $Rank(G)$.*

(b) For any nonzero $x \in G$, denote its archimedean equivalence class $[x] := v(x)$, and denote $[0] := \infty$. The map

$$\begin{array}{rccc} v: & G & \rightarrow & \Gamma \cup \{\infty\} \\ & x & \mapsto & v(x) \end{array}$$

is a valuation, which is called the ***natural valuation*** *of G.*

Definition 0.8 *Let $(\Gamma, \leq)$ be an ordered set and $\{B_\gamma, \gamma \in \Gamma\}$ be a family of archimedean abelian groups (consequently $B_\gamma \hookrightarrow (\mathbb{R}, +, \leq)$ by Hölder's theorem). The* ***ordered Hahn sum*** *is defined to be the Hahn sum $G = \overrightarrow{\coprod}_{\gamma\in\Gamma} B_\gamma$ (i.e. the direct sum from the B_γ's) endowed with the lexicographic ordering. Similarly, we define the* ***ordered Hahn product*** *$\overrightarrow{\mathbf{H}}_{\gamma\in\Gamma} B_\gamma$.*

(c) Given $x \in G$, $x \neq 0$, we put $v(x) := \gamma \in \Gamma$. Then we have

$$\begin{array}{rcl} G^\gamma := \{a \in G \mid v(a) \geq \gamma\} & = & C_x; \\ G_\gamma := \{a \in G \mid v(a) > \gamma\} & = & D_x. \end{array}$$

and consequently

$$G^\gamma/G_\gamma =: B(\gamma) = B_x := C_x/D_x$$

which is an archimedean group.

(Hint: prove that for any nonzero $x,y \in G$, we have $x \sim^+ y \Leftrightarrow C_x = C_y$ and $D_x = D_y$.)

Proof.
(a) We consider two nonzero elements $x,y \in G$ such that $x <<^+ y$. Then applying the preceding proposition, for any $a \in [x]$ and $b \in [y]$, we obtain that $a <<^+ y$ and $x <<^+ b$, which implies that $a <<^+ b$. Thus $\leq_\Gamma$ is well-defined. Moreover, for any nonzero elements $x,y \in G$, we have a trichotomy: either $x <<^+ y$, or $x \sim^+ y$, or $y <<^+ x$, which are pairwise exclusive. This means that the relation $\leq_\Gamma$ is total on Γ.
Furthermore, the relation $\leq_\Gamma$ is clearly reflexive. Consider now $[x],[y] \in \Gamma$ such that $[x] \leq_\Gamma [y]$ and $[y] \leq_\Gamma [x]$. This means that we have ($x <<^+ y$ or $x \sim^+ y$) and ($y <<^+ x$ or $x \sim^+ y$). By exclusivity of the 3 cases, it implies that $x \sim^+ y \Leftrightarrow [x] = [y]$: the relation $\leq_\Gamma$ is anti-symetric. Now the transitivity of $\leq_\Gamma$ follows directly from the transitivity of $<<^+$ and $\sim^+$, and from their compatibility (see the preceding proposition).

Thus $\leq_\Gamma$ is a total ordering on Γ.

(b) Firstly, it is clear by definition of v that $v(x) := [x] \neq \infty \Leftrightarrow x \neq 0$.
Secondly, consider $n \in \mathbb{Z}$ and $x \in G$. We have $v(nx) = [nx] = [x] = v(x)$ (indeed, by definition of the archimedean equivalence relation, we have $x \sim^+ nx$ for any $n \in \mathbb{Z}$).
Thirdly, consider $x,y \in G$. We have $v(y - x) = [y - x]$. Suppose that $x <<^+ y \Leftrightarrow v(y) < v(x)$. Without loss of generality, suppose that $y > 0$ and $x > 0$. So we have $y - x < y$, but $2(y - x) > y$ since $y - 2x > 0$. This means that $y - x \sim^+ y$, and so $v(y-x) = v(y) = \min\{v(x),v(y)\}$. Suppose now that $x \sim y \Leftrightarrow v(x) = v(y)$. If $x = y$, we trivially have $v(y - x) = \infty > \min\{v(x),v(y)\}$. If not, we may assume without loss of generality that $x < y < nx$ for some $n \in \mathbb{N}$. Then $0 < y - x < (n - 1)x$, which implies that $y - x <<^+ x$ or $y - x \sim^+ x$. Equivalently, we have $v(y - x) \leq_\Gamma v(x) = \min\{v(y),v(x)\}$.
So v is a valuation on G.

(c) Fix $x \in G$, $x \neq 0$. First, we notice that the set $\{y \in G \mid \exists n \in \mathbb{N}, n|x| \geq |y|\}$ is a convex subgroup of G which contains x. This implies that for any nonzero $y \in G$, if there exists $n \in \mathbb{N}$ such that $|y| \leq n|x|$, then $y \in C_x$. Similarly, the set $\{y \in G \mid \forall n \in \mathbb{N}, n|y| < |x|\}$ is a convex subgroup of G, which does not contain x. This implies that for any nonzero $y \in G$, if for any $n \in \mathbb{N}$, we have $n|y| < |x|$, then $y \in D_x$. By interchanging the role of x and y, we obtain that $x \sim^+ y \Leftrightarrow C_x = C_y$ and $D_x = D_y$.
Moreover, for any $y \in G$, we have $v(y) \geq v(x)$ if and only if $y <<^+ x$ or $y \sim^+ x$. This means that there exists $n \in \mathbb{N}$ such that $0 \leq |y| < n|x|$. Thus $y \in C_x$, and so $G^\gamma \subseteq C_x$. To obtain the converse, it suffices to note that in fact the set $G^\gamma = \{y \in G \mid \exists n \in \mathbb{N}, n|x| \geq |y|\}$, which is a convex subgroup of G, which contains x. So it must contain C_x. Therefore, $G^\gamma = C_x$.
Similarly, we have $v(y) > v(x)$ if and only if $y <<^+ x$. This means that for any $n \in \mathbb{N}$, we have $0 \leq n|y| < |x|$. Thus $y \in D_x$, and so $D_x \subseteq G_\gamma$. But, we have $G_\gamma = \{y \in G \mid \forall n \in \mathbb{N}, n|y| < |x|\}$ is a convex subgroup of G, which does not contain x. Therefore, we have $G_\gamma \subseteq D_x$, and so $G_\gamma = D_x$.
Now applying Proposition 0.2 and 0.4, we obtain that

$$G^\gamma/G_\gamma =: B(\gamma) = B_x := C_x/D_x$$

is an archimedian group.