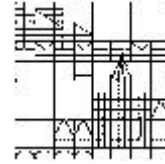


Universität Konstanz
 Fachbereich Mathematik und Statistik
 Prof. Dr. Salma Kuhlmann
 Mitarbeiter: Dr. Mickaël Matusinski
 Büroraum F 409
 mickael.matusinski@uni-konstanz.de



Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 14

These exercises will be collected Tuesday 9 February in the mailbox number 15 of the Mathematics department.

1. Consider a system of valued \mathbb{R} -vector spaces $S = [\mathbb{Q}, \{B(q); q \in \mathbb{Q}\}]$ with $B(q) \simeq \mathbb{R}$ for any $q \in \mathbb{Q}$, the corresponding Hahn sum $\prod_{\gamma \in \Gamma} B(\gamma)$ and an automorphism $\sigma : \mathbb{Q} \rightarrow \mathbb{Q}$ of the ordered set (\mathbb{Q}, \leq) . Then show that the map

$$\tilde{\sigma} : \prod_{\gamma \in \Gamma} B(\gamma) \rightarrow \prod_{\gamma \in \Gamma} B(\gamma)$$

such that $\tilde{\sigma}(s)(q) := s(\sigma(q))$ for any $s \in \prod_{\gamma \in \Gamma} B(\gamma)$ and any $q \in \mathbb{Q}$, is an automorphism of valued vector spaces.

2. **Definition 0.1** *Let $(G, +, \leq)$ be an ordered abelian group. A subgroup $C \subset G$ is said to be convex if for any $c_1, c_2 \in C$ and for any $x \in G$ such that $c_1 \leq x \leq c_2$, then $x \in C$.*

(a) Let $(G, +, \leq)$ be an ordered abelian group. Show that the set of all convex subgroups of G is totally ordered by \subseteq , contains the trivial subgroup $\{0\} \subset G$ and G , and is stable under union and intersection.

(b) Given a convex subgroup C of an ordered abelian group $(G, +, \leq)$, we define on the group $(G/C, +)$ a induced relation \leq by

$$\text{for any } x, y \in G, x \leq y \Rightarrow x + C \leq y + C.$$

Show that $(G/C, +, \leq)$ is an ordered abelian group.

Definition 0.2 Given two ordered groups (G_1, \leq) and (G_2, \leq) , and a group morphism $h : (G_1, \leq) \rightarrow (G_2, \leq)$, we say that h is an order preserving morphism if for any $x, y \in G_1$, $x \leq y \Rightarrow h(x) \leq h(y)$.

(c) With the same hypothesis as in Question (b), prove that the canonical projection $\Pi : G \rightarrow G/C$ is an order preserving morphism.

3. **Definition 0.3** • A sequence $s := (a_\rho)_{\rho \in \Lambda}$ (Λ being a well-ordered set) in a valued vector space (V, v) is said to be **pseudo-Cauchy** if for any $\rho < \sigma < \tau$, we have $v(a_\sigma - a_\rho) < v(a_\tau - a_\sigma)$.
- For any $\rho \in \Lambda$, we define $\gamma_\rho := v(a_{\rho+1} - a_\rho)$. Then the sequence $(\gamma_\rho)_{\rho \in \Lambda}$ is strictly increasing in Γ .
 - If there exists $\rho_0 \in \Lambda$ such that for any $\rho \geq \rho_0$, $v(a_\rho) = v(a_{\rho_0})$, then we define this value to be the **ultimate value** of s : $Ult(s) := v(a_{\rho_0})$.
 - An element $x \in V$ is said to be a **pseudo-limit** of a pseudo-Cauchy sequence $s := (a_\rho)_{\rho \in \Lambda}$ if $v(x - a_\rho) = \gamma_\rho$ for any $\rho \in \Lambda$.
 - The **breadth** of a pseudo-Cauchy sequence $s := (a_\rho)_{\rho \in \Lambda}$ is by definition $Br(s) := \{y \in V \mid v(y) > \gamma_\rho \ \forall \rho\}$.

Consider the ordered set $\Gamma = \mathbb{N} \cdot \mathbb{N}$ which has order type ω^2 (i.e. the set $\mathbb{N} \times \mathbb{N}$ endowed with the lexicographic order: see ÜA Blatt 13). Consider the system of ordered \mathbb{Q} -vector spaces $S := [\Gamma, \{B(\gamma); \gamma \in \Gamma\}]$ where $B(\gamma) = \mathbb{R}$ for any γ , and the corresponding Hahn sum $M := \prod_{\gamma \in \Gamma} B(\gamma)$ and Hahn product $N := \mathbf{H}_{\gamma \in \Gamma} B(\gamma)$

endowed as usual with the valuation $v := v_{\min}$.

We define the following sequences in M :

- $s^{(1)} := (a_n^{(1)})_{n \in \mathbb{N}^*}$
where for any $(k, l) \in \Gamma$, $a_n^{(1)}(k, l) := \begin{cases} l^k & \text{if } k \leq n, l \leq n \\ 0 & \text{if not} \end{cases}$
- $s^{(2)} := (a_n^{(2)})_{n \in \mathbb{N}^*}$
where for any $(k, l) \in \Gamma$, $a_n^{(2)}(k, l) := \begin{cases} n^k & \text{if } k \leq n, l = n \\ 0 & \text{if not} \end{cases}$
- $s^{(3)} := (a_n^{(3)})_{n \in \mathbb{N}^*}$
where for any $(k, l) \in \Gamma$, $a_n^{(3)}(k, l) := \begin{cases} n^n & \text{if } k = l = n \\ 0 & \text{if not} \end{cases}$

(a) Prove that these sequences are pseudo-Cauchy and compute the corresponding $\gamma_n^{(i)} := v(a_{n+1}^{(i)} - a_n^{(i)})$, $i = 1, 2, 3$.

(b) Compute the corresponding $Ult(s^{(i)})$ (whenever it is defined) and $Br(s^{(i)})$, $i = 1, 2, 3$ in M as well as in N .

(c) For each of these 3 sequences, describe the set of all pseudo-limits in M and in N .