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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 14 - Lösung

1. Consider a system of valued $\mathbb{R}$-vector spaces $S = [\mathbb{Q},\{B(q); q \in \mathbb{Q}\}]$ with $B(q) \simeq \mathbb{R}$ for any $q \in \mathbb{Q}$, the corresponding Hahn sum $\coprod_{\gamma\in\Gamma} B(\gamma)$ and an automorphism $\sigma : \mathbb{Q} \to \mathbb{Q}$ of the ordered set $(\mathbb{Q}, \leq)$.
We consider the map
$$\tilde{\sigma} : \coprod_{\gamma\in\Gamma} B(\gamma) \to \coprod_{\gamma\in\Gamma} B(\gamma)$$
such that $\tilde{\sigma}(s)(q) := s(\sigma(q))$ for any $s \in \coprod_{\gamma\in\Gamma} B(\gamma)$ and any $q \in \mathbb{Q}$.
Given any $s_1,s_2 \in \coprod_{\gamma\in\Gamma} B(\gamma)$, and $r_1,r_2 \in \mathbb{R}$, then for any $q \in \mathbb{Q}$, we have:
$$\begin{array}{rcl}\tilde{\sigma}(r_1 s_1 + r_2 s_2)(q) & := & (r_1 s_1 + r_2 s_2)(\sigma(q)) \\ & = & r_1 s_1(\sigma(q)) + r_2 s_2(\sigma(q)) \\ & = & r_1\tilde{\sigma}(s_1)(q) + r_2\tilde{\sigma}(s_2)(q).\end{array}$$
Moreover, the following map is the functional inverse $\tilde{\sigma}^{-1}$ of $\tilde{\sigma}$:
$$\tilde{\sigma}^{-1} : \coprod_{\gamma\in\Gamma} B(\gamma) \to \coprod_{\gamma\in\Gamma} B(\gamma)$$
such that $\tilde{\sigma}^{-1}(s)(q) := s(\sigma^{-1}(q))$ for any $s \in \coprod_{\gamma\in\Gamma} B(\gamma)$ and any $q \in \mathbb{Q}$, where σ^{-1} is the functional inverse of the automorphism σ. Thus $\tilde{\sigma}$ is bijective.

Consider $s \in \coprod_{\gamma\in\Gamma} B(\gamma)$. We have

$$\begin{aligned}
v(\tilde{\sigma}(s)) &= \min\{\text{support } \tilde{\sigma}(s)\} \\
&= \min\{q \in \mathbb{Q} \mid \tilde{\sigma}(s)(q) \neq 0\} \\
&= \sigma \min\{q \in \mathbb{Q} \mid \ s(\sigma(q)) \neq 0\} \\
&= \min\{\sigma^{-1}(q') \in \mathbb{Q} \mid \ s(q') \neq 0\} \\
&= \sigma^{-1}(\min\{\text{support } s\}) \\
&= \sigma^{-1}(v(s))
\end{aligned}$$

Since $\sigma^{-1} : \mathbb{Q} \to \mathbb{Q}$ is an automorphism of the ordered set $\mathbb{Q}$, we get that $\tilde{\sigma}$ is value preserving and therefore is an automorphism of valued vector spaces.

2. **Definition 0.1** *Let $(G, +, \leq)$ be an ordered abelian group. A subgroup $C \subset G$ is said to be convex if for any $c_1, c_2 \in C$ and for any $x \in G$ such that $c_1 \leq x \leq c_2$, then $x \in C$.*

(a) Let $(G, +, \leq)$ be an ordered abelian group, and C_1, C_2 be two convex subgroups. Suppose for instance that there exists $c_1 \in C_1 \setminus C_2$. Since C_2 is convex, then either $c_1 < C_2$ or $C_2 < c_1$. Since G is an ordered abelian group, it implies that $c_1 < C_2 < -c_1$ or respectively $-c_1 < C_2 < c_1$. But since C_1 is convex, for any $c_2 \in C_2$, the inequalities $-c_1 < c_2 < c_1$ imply that $c_2 \in C_1$. Thus $C_2 \subseteq C_1$.
The group G itself as well as the trivial group $\{0\}$ are clearly convex subgroups of G.
Consider two convex subgroups C_1, C_2 of G. We know that either $C_1 \subseteq C_2$, or $C_2 \subseteq C_1$. Suppose for instance that $C_2 \subseteq C_1$. Then $C_1 \cap C_2 = C_2$ and $C_1 \cup C_2 = C_1$ which are convex subgroups of G.

(b) Given a convex subgroup C of an ordered abelian group $(G, +, \leq)$, we define on the group $(G/C, +)$ a relation $\leq$ by

$$\text{for any } x, y \in G,\ x \leq y \Rightarrow x + C \leq y + C.$$

The fact that $(G/C, +)$ is an abelian group follows directly from the fact that the relation

$$\forall x, y \in G,\ x \sim_C y \Leftrightarrow x - y \in C$$

is a congruence relation, i.e. an equivalence relation compatible with the addition (if $x_1 \sim_C x_2$ and $y_1 \sim_C y_2$, then $x_1 + x_2 \sim_C y_1 + y_2$).

We notice that $x + C \leq y + C$ implies that either $x + C = y + C \Leftrightarrow x - y \in C$, or $x + c_1 < y + c_2$ for any $c_1, c_2 \in C$. The fact that $(G/C, \leq)$ is an ordered set follows directly from the definition of the ordering and the fact that $(G, \leq)$ is totally ordered (check the axioms of reflexivity, anti-symmetry and transitivity).

Concerning the fact that $(G/C, \leq)$ is a totally ordered set, consider $x + C$ and $y + C$ in G/C. Then either $x - y \in C$ which implies that $x + C = y + C$, or we have $x + c_1 < y + c_2$ or $x + c_1 > y + c_2$ for any $c_1, c_2 \in C$ (since C is convex). In these last 2 cases, we have $x < y$ or respectively $x > y$ (since $0 \in C$ as a subgroup of

G) which implies that $x + C < y + C$, respectively $x + C > y + C$.

To show that $(G/C, +, \leq)$ is an ordered abelian group, take any $x+C \leq y+C$ and $z+C$ in G/C. Then $x+C \leq y+C$ implies that either $x+C = y+C \Leftrightarrow x-y \in C$, or $x < y$. So we have either $(x+z)+C = (y+z)+C$, or $x+z < y+z$ in G. But this last inequality implies that $(x+z)+C \leq (y+z)+C$ in G/C.

Definition 0.2 *Given two ordered groups $(G_1, \leq)$ and $(G_2, \leq)$, and a group morphism $h : (G_1, \leq) \to (G_2, \leq)$, we say that h is an order preserving morphism if for any $x,y \in G_1$, $x \leq y \Rightarrow h(x) \leq h(y)$.*

(c) The canonical projection $\Pi : G \to G/C$ is a group morphism. Now, take $x \leq y$ in G. This implies by definition of the ordering on G/C that $\Pi(x) = x + C \leq y + C = \Pi(y)$ in G/C. The canonical projection is order preserving.

3. **Definition 0.3** • *A sequence $s := (a_\rho)_{\rho\in\Lambda}$ (Λ being a well-ordered set) in a valued vector space (V,v) is said to be **pseudo-Cauchy** if for any $\rho < \sigma < \tau$, we have $v(a_\sigma - a_\rho) < v(a_\tau - a_\sigma)$.*
   • *For any $\rho \in \Lambda$, we define $\gamma_\rho := v(a_{\rho+1} - a_\rho)$. Then the sequence $(\gamma_\rho)_{\rho\in\Lambda}$ is strictly increasing in Γ.*
   • *If there exists $\rho_0 \in \Lambda$ such that for any $\rho \geq \rho_0$, $v(a_\rho) = v(a_{\rho_0})$, then we define this value to be the **ultimate value** of s: $Ult(s) := v(a_{\rho_0})$.*
   • *An element $x \in V$ is said to be a **pseudo-limit** of a pseudo-Cauchy sequence $s := (a_\rho)_{\rho\in\Lambda}$ if $v(x - a_\rho) = \gamma_\rho$ for any $\rho \in \Lambda$.*
   • *The **breadth** of a pseudo-Cauchy sequence $s := (a_\rho)_{\rho\in\Lambda}$ is by definition $Br(s) := \{y \in V \mid v(y) > \gamma_\rho \ \forall\rho\}$.*

Consider the ordered set $\Gamma = \mathbb{N}.\mathbb{N}$ which has order type ω^2 (i.e. the set $\mathbb{N} \times \mathbb{N}$ endowed with the lexicographic order: see ÜA Blatt 13). Consider the system of ordered $\mathbb{Q}$-vector spaces $S := [\Gamma, \{B(\gamma); \gamma \in \Gamma\}]$ where $B(\gamma) = \mathbb{R}$ for any γ, and the corresponding Hahn sum $M := \coprod_{\gamma\in\Gamma} B(\gamma)$ and Hahn product $N := \mathbf{H}_{\gamma\in\Gamma} B(\gamma)$ endowed as usual with the valuation $v := v_{\min}$.
We define the following sequences in M:

- $s^{(1)} := (a_n^{(1)})_{n\in\mathbb{N}^*}$
  where for any $(k,l) \in \Gamma$, $a_n^{(1)}(k,l) := \begin{cases} l^k & \text{if } k \leq n, l \leq n \\ 0 & \text{if not} \end{cases}$
- $s^{(2)} := (a_n^{(2)})_{n\in\mathbb{N}^*}$
  where for any $(k,l) \in \Gamma$, $a_n^{(2)}(k,l) := \begin{cases} n^k & \text{if } k \leq n, l = n \\ 0 & \text{if not} \end{cases}$
- $s^{(3)} := (a_n^{(3)})_{n\in\mathbb{N}^*}$
  where for any $(k,l) \in \Gamma$, $a_n^{(3)}(k,l) := \begin{cases} n^n & \text{if } k = l = n \\ 0 & \text{if not} \end{cases}$

(a) For any $p < q \in \mathbb{N}^*$, we have

$$\text{for any } (k,l) \in \Gamma,\ a_q^{(1)}(k,l) - a_p^{(1)}(k,l) = \left| \begin{array}{ll} l^k & \text{if } (k \le p \text{ and } p+1 \le l \le q) \text{ or } (p+1 \le k \le q \text{ and } l \le q) \\ 0 & \text{if not.} \end{array} \right.$$

So, for any $n < p < q \in \mathbb{N}^*$, we compute

$$v(a_p^{(1)} - a_n^{(1)}) = (1,n+1) < (1,p+1) = v(a_q^{(1)} - a_p^{(1)}).$$

The sequence $s^{(1)}$ is pseudo-Cauchy.
Moreover, we have $\gamma_n^{(1)} = (1,n+1)$.

For any $p < q \in \mathbb{N}^*$, we have

$$\text{for any } (k,l) \in \Gamma,\ a_q^{(3)}(k,l) - a_p^{(3)}(k,l) = \left| \begin{array}{ll} -p^k & \text{if } k \le p \text{ and } l = p \\ q^k & \text{if } k \le q \text{ and } l = q \\ 0 & \text{if not.} \end{array} \right.$$

So, for any $n < p < q \in \mathbb{N}^*$, we compute

$$v(a_p^{(2)} - a_n^{(2)}) = (1,n) < (1,p) = v(a_q^{(2)} - a_p^{(2)}).$$

The sequence $s^{(2)}$ is pseudo-Cauchy.
Moreover, we have $\gamma_n^{(2)} = (1,n)$.

For any $p < q \in \mathbb{N}^*$, we have

$$\text{for any } (k,l) \in \Gamma,\ a_q^{(3)}(k,l) - a_p^{(3)}(k,l) = \left| \begin{array}{ll} -p^p & \text{si } k = l = p \\ q^q & \text{si } k = l = q \\ 0 & \text{sinon.} \end{array} \right.$$

So, for any $n < p < q \in \mathbb{N}^*$, we compute

$$v(a_p^{(3)} - a_n^{(3)}) = (n,n) < (p,p) = v(a_q^{(3)} - a_p^{(3)}).$$

The sequence $s^{(3)}$ is pseudo-Cauchy.
Moreover, we have $\gamma_n^{(1)} = (n,n)$.

(b) The value $Ult(s^{(i)})$ is only defined in the case $i = 1$, and we have $Ult(s^{(1)}) = v(a_1^{(1)}) = (1,1)$.
We also have in M:

$$\begin{array}{lll} \bullet\ Br_M(s^{(1)}) & = & \coprod_{\gamma \in \mathbb{N}^*.\mathbb{N} \subsetneq \Gamma} B(\gamma) \\ \bullet\ Br_M(s^{(2)}) & = & \coprod_{\gamma \in \mathbb{N}^*.\mathbb{N} \subsetneq \Gamma} B(\gamma) \\ \bullet\ Br_M(s^{(3)}) & = & \{0\}. \end{array}$$

and in N:

$$\begin{array}{lll} \bullet\ Br_N(s^{(1)}) & = & \mathbf{H}_{\gamma \in \mathbb{N}^*.\mathbb{N} \subsetneq \Gamma} B(\gamma) \\ \bullet\ Br_N(s^{(2)}) & = & \mathbf{H}_{\gamma \in \mathbb{N}^*.\mathbb{N} \subsetneq \Gamma} B(\gamma) \\ \bullet\ Br_N(s^{(3)}) & = & \{0\}. \end{array}$$

(c) Given a pseudo-Cauchy sequence $s := (a_\rho)_{\rho \in \Lambda}$ and two pseudo-limits x and y, we have $x - y \in Br(s)$. So, in order to obtain all the pseudo-limits of a given pseudo-Cauchy sequence, it suffices to know one particular pseudo-limit x_0 and the breadth $Br(s)$ of the sequence.

For the sequence $s^{(1)}$, a pseudo-limit $x_0^{(1)}$ has to contain $(1,n)$ for all $n \in \mathbb{N}$ in its support (otherwise we would not have $v(x_0^{(1)} - a_n^{(1)}) = \gamma_n = (1,n+1)$). Thus it cannot be an element of M. So the set of pseudo-limits of $s^{(1)}$ in M is empty.
A pseudo-limit in N is given for instance by

$$x_0^{(1)}(k,l) := \left| \begin{array}{lll} l & \text{if} & k = 1, l \in \mathbb{N} \\ 0 & \text{if not} & \end{array} \right.$$

Thus the set of all pseudo-limits of $s^{(1)}$ in N is given by

$$x^{(1)} + Br_N(s^{(1)}) = x^{(1)} + \mathbf{H}_{\gamma \in \mathbb{N}^*.\mathbb{N} \subsetneq \Gamma} B(\gamma).$$

For the sequence $s^{(2)}$, since $v(a_n^{(2)}) = (1,n)$ is strictly increasing as n increases, $x_0^{(2)} = 0$ is a pseudo-limit of $s^{(2)}$ in M as well as in N. So the set of pseudo-limits in M is

$$x_0^{(2)} + Br_M(s^{(2)}) = \coprod_{\gamma \in \mathbb{N}^*.\mathbb{N} \subsetneq \Gamma} B(\gamma).$$

The set of pseudo-limits in N is

$$x_0^{(2)} + Br_N(s^{(2)}) = \mathbf{H}_{\gamma \in \mathbb{N}^*.\mathbb{N} \subsetneq \Gamma} B(\gamma).$$

For the sequence $s^{(3)}$, since $v(a_n^{(3)}) = (n,n)$ is strictly increasing as n increases, $x_0^{(3)} = 0$ is a pseudo-limit of $s^{(3)}$ in M as well as in N. So the set of pseudo-limits in M as well as in N is

$$x_0^{(3)} + Br_M(s^{(3)}) = \{0\} = x_0^{(3)} + Br_N(s^{(3)}).$$