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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 12

These exercises will be collected Tuesday 26 January in the mailbox number 15 of the Mathematics department.

Definition 0.1 *Let R be a real closed field. Let $A \subseteq R^n$ be a semi-algbraic set.*
(i) A ***semialgebraic path*** *in A is a continuous semialgebraic map $\alpha : (0,1) \to A$.*
(ii) The set A is ***semialgebraically compact*** *if for every path $\alpha : (0,1) \to A$, $\lim_{t\to 0^+} \alpha(t)$ exists and is in A.*

1. Prove the following theorem:

   Theorem 0.2 (semialgebraic choice = Semialgebraische Auswahl) *Let A and B be semialgebraic sets and $f : A \to B$ be a surjective semialgebraic map. Then f has a semialgebraic section, i.e. there is a semialgebraic map $g : B \to A$ with $f(g(y)) = y$ for any $y \in B$.*

   (Hint: use Exercise 4 in Blatt 11.)

2. Prove the following Corollary from Lecture 21:

   Corollary 0.3 (Curve Selection Lemma: unbounded case) *Let $A \subseteq R^n$ be an unbounded semialgebraic set. Then there exists a semialgebraic path $\alpha :]0,1[\to A$ with $\lim_{t\to 0} \|\alpha(t)\| = +\infty$.*

   (Hint: use the results about stereographic projection obtained in Exercise 4 of Blatt 10.)

3. (a) The following Lemma has been used in Lecture 21 to prove the theorem on the image of a semialgebraically compact set by a semialgebraic map. Prove it:

   Lemma 0.4 *Let A and B be semialgebraic sets and $f : A \to B$ be a semialgebraic map. Let $\beta :]0,1[\to B$ be a semialgebraic path in B with $\beta(]0,1[) \subseteq f(A)$. Then there exists $c \in R$ with $0 < c < 1$ and there exists a semialgebraic path $\alpha :]0,c[\to A$ such that $\beta(t) = f(\alpha(t))$ for any $t \in]0,c[$.*

   (Hint: use the result of Exercise 2.(b) of Blatt 11.)

   (b) Let A be a semialgebraically compact set. Deduce that any semialgebraic function $f : A \to R$ has a minimum and a maximum.

4. (a) Let $A \subseteq R^n$ be a semialgebraic set, $x \in A$. Show that there exist a neighborhood U of x in R^n and a nonnegative integer d such that, for every semialgebraic neighborhood $V \subset U$ of x in R^n, $\dim(V \cap A) = d$.

   The integer d is called the **dimension of** A **at** x and is denoted by $\dim_x A$.

   (b) Show that

   $$\dim A = \max\{\dim_x A; x \in A\}.$$

   (c) Show that $\{x \in A; \dim_x A = \dim A\}$ is a closed semialgebraic subset of A.