Universität Konstanz
Fachbereich Mathematik und Statistik
Prof. Dr. Salma Kuhlmann
Mitarbeiter: Dr. Mickaël Matusinski
Büroraum F 409
mickael.matusinski@uni-konstanz.de

Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 10

These exercises will be collected Tuesday 12 January in the mailbox number 15 of the Mathematics department.

Theorem 0.1 (Cell Decomposition = Zell Zerlegung) *Let R be a real closed field. Any semi-algebraic subset $A \subset R^n$ is the disjoint union of a finite number of semi-algebraic sets, each of them semi-algebraically homeomorphic to an open hypercube $]0,1[^d \subset R^d$, for some $d \in \mathbb{N}$ (with $]0,1[^0$ being a point).*

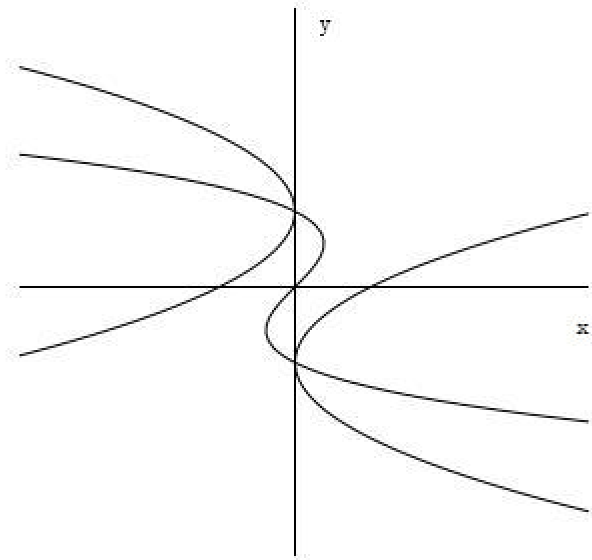
1. This exercise concerns the proof of this **Cell Decomposition Theorem**, which is done by induction on $n \in \mathbb{N}$. Concerning the induction step, one considers a semi-algebraic subset $A \subset R^{n+1}$ and the polynomials $f_1(\underline{X},Y),\dots,f_s(\underline{X},Y)$ of $R[\underline{X},Y]$ which define A. The proof is done showing that there exists a **slicing** $(A_i,\{\xi_{i,j},\ j = 1,\dots,l_i\})_{i=1,\dots,m}$ of the family $f_1(\underline{X},Y),\dots,f_s(\underline{X},Y)$ with respect to the variable Y. Our purpose here is to clarify:
   - the role in this proof of adding the derivatives with respect to Y to the family $f_1(\underline{X},Y),\dots,f_s(\underline{X},Y)$;
   - how we can remove the roots $\xi_{i,j}(\underline{X})$ coming from these new polynomials and obtain the right slicing for the initial family.

   Consider the following two-variables polynomial
   $$f(X,Y) = (X + (Y-1)^2)^2(X-(Y+1)^2)^2$$
   of $R[X,Y]$ and the corresponding semi-algebraic subset of R^2
   $$A := \{(x,y) \in R^2 \mid f(x,y) = 0\}.$$
   (a) For any $x \in R$, give the two roots of $f(x,Y)$ and deduce its sign matrix with respect to x.

   (b) Draw $A \subset R^2$ and deduce that we cannot find two *continuous* semi-algebraic functions $\xi_1(x) < \xi_2(x)\ :\ R \to R$ so that we have a slicing $(A_1 = R,\{\xi_1(x) < \xi_2(x)\})$ of A.

   The semi-algebraic subset
   $$\tilde{A} := \{(x,y) \in R^2 \mid f(x,y) = 0 = f'(x,y)\}$$

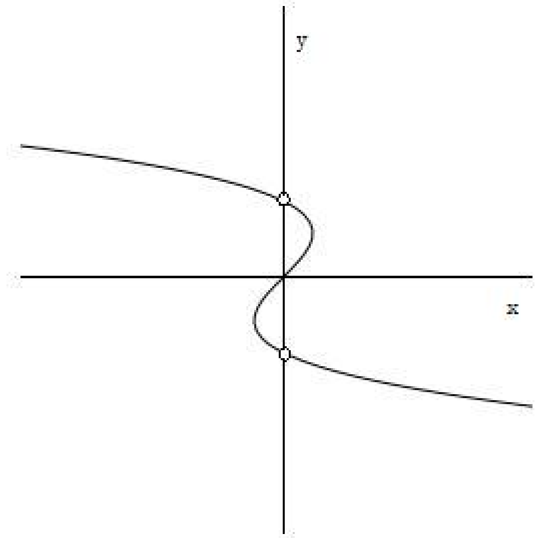
of R^2 can be represented as



(c) Compute $f'(X,Y)$ the derivative with respect to Y of $f(X,Y)$ and compute the sign matrices $\mathrm{Sign}_R(f(x,Y),f'(x,Y))$ with respect to $x \in R$.
(*Hint: recall that the cubic polynomial* $y^3 - y + x$ *has 1, 2 or 3 roots, whenever its discriminant* $\Delta := x^2 - \frac{4}{27}$ *is* $> 0, = 0, < 0$ *respectively, and use the preceding picture to order all the roots.*)

(d) Deduce the slicing $(\tilde{A}_i, \{\tilde{\xi}_{i,j},\ j = 1, \ldots, l_i\})_{i=1,\ldots,m}$ of $\tilde{A}$.

(e) Show that we can only remove from the precedingly computed slicing, the $\tilde{\xi}_{i,j}$'s such that the union of their graphs $\bigcup_{i,j} \Gamma(\tilde{\xi}_{i,j})$ is the following curve $\{(x,y) \in R^2 \mid y^3 - y + x = 0\}$ minus the 2 points indicated for $x = 0$:



(f) Conclude that the slicing of f is given by $(A_i,\{\xi_{i,1} < \xi_{i,2}\})_{i=1,2,3}$ with $A_1 =]-\infty,0[$, $A_2 = \{0\}$ and $A_3 =]0,\infty[$. Give the formulas for the $\xi_{i,j}$'s.

2. Let $d \in \mathbb{N}$. Show that the semi-algebraic sets

$$R^d,\ ]0,\infty[^d,\ ]0,1[^d \text{ and } B_d(\underline{0},1) := \{\underline{x} \in R^d \mid \|\underline{x}\| < 1\}$$

are pairwise **semi-algebraically homeomorph**. Such semi-algebraic sets are called **cells** (**Zell**)

3. Let $A \subset R^n$ be semi-algebraic. Show that:

(a) for any $\underline{x} \in R^n$, the expression $\text{dist}(\underline{x},A) := \inf\{\|\underline{x}-\underline{y}\| \mid \underline{y} \in A\}$ is well defined;

(b) the map

$$\begin{array}{rccc} \text{dist}: & R^n & \to & R \\ & \underline{x} & \mapsto & \text{dist}(\underline{x},A) \end{array}$$

is semi-algebraic, continuous, vanishes on $\text{Clos}(A)$ and is positive elsewhere.

4. Let $n \in \mathbb{N}$, $S_n(\underline{0},1) := \{\underline{x} \in R^{n+1} \mid \|\underline{x}\| = 1\}$ be the n-hypersphere, and $\infty := (1,0,\dots,0)$ its north pole. Show that:

(a) the **stereographic projection** $p : S_n(\underline{0},1) \setminus \{\infty\} \to R^n$ is a semi-algebraic homeomorphism;

(b) a subset of $S \subset R^n$ is unbounded if and only if the closure of p^{-1} contains ∞.