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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 10 - Solution

Theorem 0.1 (Cell Decomposition = Zell Zerlegung) *Let R be a real closed field. Any semi-algebraic subset $A \subset R^n$ is the disjoint union of a finite number of semi-algebraic sets, each of them semi-algebraically homeomorphic to an open hypercube $]0,1[^d \subset R^d$, for some $d \in \mathbb{N}$ (with $]0,1[^0$ being a point).*

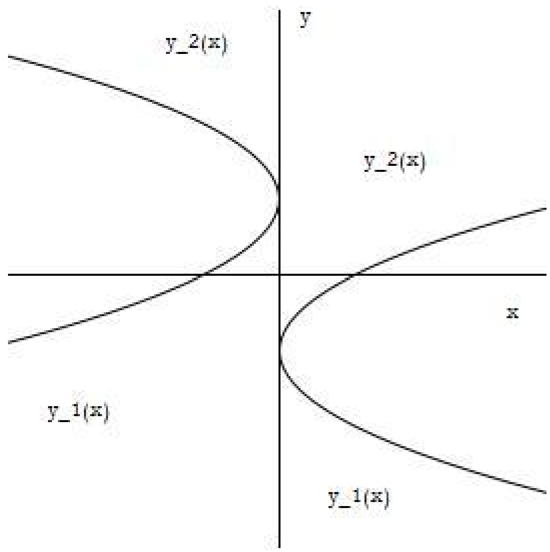
1. This exercise concerns the proof of this **Cell Decomposition Theorem**, which is done by induction on $n \in \mathbb{N}$. Concerning the induction step, one considers a semi-algebraic subset $A \subset R^{n+1}$ and the polynomials $f_1(\underline{X},Y),\ldots,f_s(\underline{X},Y)$ of $R[\underline{X},Y]$ which define A. The proof is done showing that there exists a **slicing** $(A_i,\{\xi_{i,j},\ j=1,\ldots,l_i\})_{i=1,\ldots,m}$ of the family $f_1(\underline{X},Y),\ldots,f_s(\underline{X},Y)$ with respect to the variable Y. Our purpose here is to clarify:
   - the role in this proof of adding the derivatives with respect to Y to the family $f_1(\underline{X},Y),\ldots,f_s(\underline{X},Y)$;
   - how we can remove the roots $\xi_{i,j}(\underline{X})$ coming from these new polynomials and obtain the right slicing for the initial family.

   Consider the following two-variables polynomial
   $$f(X,Y)=(X+(Y-1)^2)^2(X-(Y+1)^2)^2$$
   of $R[X,Y]$ and the corresponding semi-algebraic subset of R^2
   $$A:=\{(x,y)\in R^2 \mid f(x,y)=0\}.$$
   (a)
   - If $x>0$, the two roots of $f(x,Y)$ are
   $$y_1(x)=-\sqrt{x}+1 \text{ and } y_2(x)=\sqrt{x}+1.$$
   - If $x=0$, the two roots of $f(x,Y)$ are
   $$y_1(0)=-1 \text{ and } y_2(x)=1.$$
   - If $x<0$, the two roots of $f(x,Y)$ are
   $$y_1(x)=-\sqrt{-x}-1 \text{ and } y_2(x)=\sqrt{-x}-1.$$

   Note that for any x, we have $y_1(x)<y_2(x)$, and for any $x,y\in R$, $f(x,y)\geq 0$. So, for any $x\in R$, the sign matrix of $f(x,y)$ is
   $$\begin{array}{cc} y & \begin{array}{ccccc} I_0 & y_1(x) & I_1 & y_2(x) & I_2 \end{array} \\ \mathrm{Sign}_R(f(x,y)) = & \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \end{pmatrix}. \end{array}$$

Since the sign matrix is constant with respect to x, we may *a priori* have a slicing $(A_1 = R, \{\xi_1(x) < \xi_2(x)\})$ of f where $\xi_j(x) := y_j(x)$ for $j = 1,2$.

(b) The picture of $A := \{(x,y) \in R^2 \mid f(x,y) = 0\}$ is

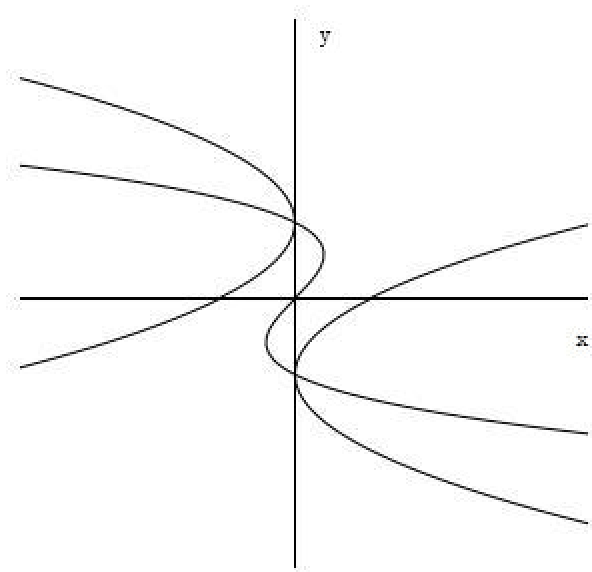


The functions y_1 and y_2 are discontinuous at 0. So $(R, \{\xi_1(x) < \xi_2(x)\})$ with $\xi_j(x) := y_j(x)$ for $j = 1,2$ is not a slicing of f.

The semi-algebraic subset

$$\tilde{A} := \{(x,y) \in R^2 \mid f(x,y) = 0 = f'(x,y)\}$$

of R^2 can be represented as



(c) The derivative with respect to Y of $f(X,Y)$ is

$$f'(X,Y) = -8(X + (Y-1)^2)(X - (Y+1)^2)(Y^3 - Y + X).$$

For any $x \in R$, the discriminant of the cubic polynomial $Y^3 - Y + x$ is $\Delta := x^2 - \frac{4}{27}1^3$. We have 3 cases:

- if $\Delta < 0 \Leftrightarrow -\sqrt{\frac{4}{27}} < x < \sqrt{\frac{4}{27}}$, the cubic polynomial $Y^3 - Y + x$ has 3 roots $y_3(x) < y_4(x) < y_5(x)$ and 2 sign changes.
- if $\Delta = 0 \Leftrightarrow x = \pm\sqrt{\frac{4}{27}}$, the cubic polynomial $Y^3 - Y + x$ has 2 roots $y_3(x) < y_4(x)$ and 1 sign change. For $x = -\sqrt{\frac{4}{27}}$, the sign change is at $y_1(x)$, and for $x = \sqrt{\frac{4}{27}}$, the sign change is at $y_2(x)$
- if $\Delta > 0 \Leftrightarrow x < -\sqrt{\frac{4}{27}}$ or $x > \sqrt{\frac{4}{27}}$, the cubic polynomial $Y^3 - Y + x$ has 2 roots $y_3(x) < y_4(x)$ and no sign change. It is > 0 whenever $x < -\sqrt{\frac{4}{27}}$ and < 0 whenever $x > \sqrt{\frac{4}{27}}$.

We obtain the following cases:

- if $x < -\sqrt{\frac{4}{27}}$, then $y_1(x) = -\sqrt{-x} + 1 < y_3(x) < y_2(x) = \sqrt{-x} + 1$ and we have

$$\mathrm{Sign}_R(f,f') = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

– if $x = -\sqrt{\frac{4}{27}} = -\frac{2}{3\sqrt{3}}$, then $y_3(x) = \frac{-1}{\sqrt{3}} < y_1(x) = -\sqrt{\frac{2}{3\sqrt{3}}} + 1 < y_4(x) = \frac{2}{\sqrt{3}} < y_2(x) = \sqrt{\frac{2}{3\sqrt{3}}} + 1$ and we have

$$\mathrm{Sign}_R(f,f') = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

– if $-\sqrt{\frac{4}{27}} < x < 0$, then $y_3(x) < y_4(x) < y_1(x) = -\sqrt{-x} + 1 < y_5(x) < y_2(x) = \sqrt{-x} + 1$ and we have

$$\mathrm{Sign}_R(f,f') = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

– if $x = 0$, then $y_3(0) = y_1(0) = -1 < y_4(0) = 0 < y_2(0) = y_5(0) = 1$ and we have

$$\mathrm{Sign}_R(f,f') = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

– if $0 < x < \sqrt{\frac{4}{27}}$, then $y_1(x) = -\sqrt{x} - 1 < y_3(x) < y_2(x) = \sqrt{x} - 1 < y_4(x) < y_5(x)$ and we have

$$\mathrm{Sign}_R(f,f') = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

– if $x = \sqrt{\frac{4}{27}} = \frac{2}{3\sqrt{3}}$, then $y_1(x) = -\sqrt{\frac{2}{3\sqrt{3}}} - 1 < y_3(x) = \frac{-2}{\sqrt{3}} < y_2(x) = \sqrt{\frac{2}{3\sqrt{3}}} - 1 < y_4(x) = \frac{1}{\sqrt{3}}$ and we have

$$\mathrm{Sign}_R(f,f') = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

– if $x > \sqrt{\frac{4}{27}}$, then $y_1(x) = -\sqrt{x} - 1 < y_3(x) < y_2(x) = \sqrt{x} - 1$ and we have

$$\mathrm{Sign}_R(f,f') = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

(d) The slicing of $\tilde{A}$ is:

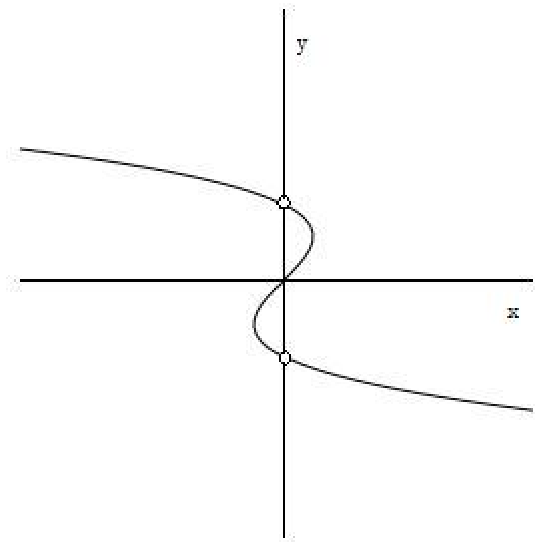
– the interval $\tilde{A}_1 = \left]-\infty, -\sqrt{\frac{4}{27}}\right[$ and the maps $\{\tilde{\xi}_{1,1}(x) = y_1(x) = -\sqrt{-x} + 1 < \tilde{\xi}_{1,2}(x) = y_3(x) < \tilde{\xi}_{1,3}(x) = y_2(x) = \sqrt{-x} + 1\}$;

– the singleton $\tilde{A}_2 = \left\{-\sqrt{\frac{4}{27}}\right\}$ and the maps $\{\tilde{\xi}_{2,1}(x) = y_3(x) = \frac{-1}{\sqrt{3}} <$

$\tilde{\xi}_{2,2}(x) = y_1(x) = -\sqrt{\frac{2}{3\sqrt{3}}} + 1 < \tilde{\xi}_{2,3}(x) = y_4(x) = \frac{2}{\sqrt{3}} < \tilde{\xi}_{2,4}(x) = y_2(x) = \sqrt{\frac{2}{3\sqrt{3}}} + 1\}$;

- the interval $\tilde{A}_3 = \left]-\sqrt{\frac{4}{27}},0\right[$ and the maps $\{\tilde{\xi}_{3,1}(x) = y_3(x) < \tilde{\xi}_{3,2}(x) = y_4(x) < \tilde{\xi}_{3,3}(x) = y_1(x) = -\sqrt{-x} + 1 < \tilde{\xi}_{3,4}(x) = y_5(x) < \tilde{\xi}_{3,5}(x) = y_2(x) = \sqrt{-x} + 1\}$;
- the singleton $\tilde{A}_4 = \{0\}$ and the maps $\{\tilde{\xi}_{4,1}(0) = y_1(0) = y_3(0) = -1 < \tilde{\xi}_{4,2}(x) = y_4(0) = 0 < \tilde{\xi}_{4,3}(0) = y_2(x) = y_5(0) = 1\}$;
- the interval $\tilde{A}_5 = \left]0,\sqrt{\frac{4}{27}}\right[$ and the maps $\{\tilde{\xi}_{5,1}(x) = y_1(x) = -\sqrt{x} - 1 < \tilde{\xi}_{5,2}(x) = y_3(x) < \tilde{\xi}_{5,3}(x) = y_2(x) = \sqrt{x} - 1 < \tilde{\xi}_{5,4}(x) = y_4(x) < \tilde{\xi}_{5,5}(x) = y_5(x)\}$;
- the singleton $\tilde{A}_6 = \left\{\sqrt{\frac{4}{27}}\right\}$ and the maps $\{\tilde{\xi}_{6,1}(x) = y_1(x) = -\sqrt{\frac{2}{3\sqrt{3}}} - 1 < \tilde{\xi}_{6,2}(x) = y_3(x) = \frac{-2}{\sqrt{3}} < \tilde{\xi}_{6,3}(x) = y_2(x) = \sqrt{\frac{2}{3\sqrt{3}}} - 1 < \tilde{\xi}_{6,4}(x) = y_4(x) = \frac{1}{\sqrt{3}}\}$;
- the interval $\tilde{A}_7 = \left]\sqrt{\frac{4}{27}},\infty\right[$ and the maps $\{\tilde{\xi}_{7,1}(x) = y_1(x) = -\sqrt{x} - 1 < \tilde{\xi}_{7,2}(x) = y_3(x) < \tilde{\xi}_{7,3}(x) = y_2(x) = \sqrt{x} - 1\}$.

(e) Note that, for each A_i, we have either $\Gamma(\tilde{\xi}_{i,j}) \subset A$ or $\Gamma(\tilde{\xi}_{i,j}) \cap A = \emptyset$. We can only remove the $\tilde{\xi}_{i,j}$'s coming *properly* from $f'(x,y)$, i.e. the parts for which A and $\tilde{A}$ do not coincide. Thus we can remove the functions $\tilde{\xi}_{1,2}$, $\tilde{\xi}_{2,1}$, $\tilde{\xi}_{2,3}$, $\tilde{\xi}_{3,1}$, $\tilde{\xi}_{3,2}$, $\tilde{\xi}_{3,4}$, $\tilde{\xi}_{4,2}$, $\tilde{\xi}_{5,2}$, $\tilde{\xi}_{5,4}$, $\tilde{\xi}_{5,5}$, $\tilde{\xi}_{6,2}$, $\tilde{\xi}_{6,4}$, $\tilde{\xi}_{7,2}$, which correspond to the following curve $\{(x,y) \in R^2 \mid y^3 - y + x = 0\}$ minus the 2 indicated points for $x = 0$:



(f) The slicing of A is given by:

- the interval $A_1 =]-\infty,0[$ and the maps $\left\{\xi_{1,1}(x) = y_1(x) = -\sqrt{-x} + 1 < \xi_{1,2}(x) = y_2(x) = \sqrt{-x} + 1\right\}$;
- the singleton $A_2 = \{0\}$ and the maps $\{\xi_{2,1}(0) = y_1(0) = -1 < \xi_{2,2}(0) = y_2(0) = 1\}$;
- the interval $A_3 =]0,\infty[$ and the maps $\left\{\xi_{3,1}(x) = y_1(x) = -\sqrt{x} - 1 < \xi_{3,2}(x) = y_2(x) = \sqrt{x} - 1\right\}$.

2. Let $d \in \mathbb{N}$. Consider the following semi-algebraic homeomorphisms:

$$\bullet F: \begin{array}{ccc} R^d & \to &]0,1[^d \\ (x_1,\dots,x_d) & \mapsto & (f(x_1),\dots,f(x_d)) \end{array}$$

where

$$f: \begin{array}{ccc} R & \to &]0,1[\\ x & \mapsto & \dfrac{x+\sqrt{1+x^2}}{2\sqrt{1+x^2}}. \end{array}$$

$$\bullet G: \begin{array}{ccc}]0,1[^d & \to &]0,+\infty[^d \\ (x_1,\dots,x_d) & \mapsto & (g(x_1),\dots,g(x_d)) \end{array}$$

where

$$g: \begin{array}{ccc}]0,1[& \to &]0,+\infty[\\ x & \mapsto & \dfrac{x}{1-x}. \end{array}$$

$$\bullet H: \begin{array}{ccc} R^d & \to & B_d(\underline{0},1) \\ (x_1,\dots,x_d) & \mapsto & (\dfrac{1}{1+\|\underline{x}\|}x_1,\dots,\dfrac{1}{1+\|\underline{x}\|}x_d) \end{array}$$

where

$$\|\underline{x}\| = \|(x_1,\dots,x_d)\| = \sqrt{x_1^2+\cdots+x_d^2}.$$

3. Let $A \subset R^n$ be semi-algebraic.:

(a) for any $\underline{x} \in R^n$, the set $\{||\underline{x}-\underline{y}|| \mid \underline{y} \in A\}$ is the image of A by the semi-algebraic function $\underline{y} \mapsto ||\underline{x}-\underline{y}||$. So it is semi-algebraic in R, which implies that it is a finite union of points and open intervals of R. Moreover it is bounded from below by 0. So the infimum is well-defined in R.

(b) The graph of the function $dist$ is

$$\Gamma(dist) = \{(\underline{x},t) \in R^{n+1} \mid (t \geq 0) \text{ and } (\forall \underline{y} \in A, t^2 \leq ||\underline{x}-\underline{y}||^2) \text{ and } (\forall \epsilon \in R, \epsilon > 0 \Rightarrow \exists \underline{y} \in A, t^2 + \epsilon > ||\underline{x}-\underline{y}||^2)\},$$

which is semi-algebraic. Moreover the function $dist$ is continuous as composition of continuous functions. It clearly vanishes on Clos(A) and is positive elsewhere.

4. Let $n \in \mathbb{N}$, $S_n(\underline{0},1) := \{\underline{x} \in R^{n+1} \mid ||\underline{x}|| = 1\}$ be the n-hypersphere, and $\infty := (1,0,\ldots,0)$ its north pole. Show that:

(a) the stereographic projection is the following application

$$\begin{array}{rccc} p: & S_n(\underline{0},1) \setminus \{\infty\} & \rightarrow & R^n \\ & (x_0,\ldots,x_n) & \mapsto & (\frac{2}{2-x_0}x_1,\ldots,\frac{2}{2-x_0}x_n) \end{array}$$

which is clearly a semi-algebraic homeomorphism;

(b) A subset of $S \subset R^n$ is unbounded if and only if it contains a sequence of points $(\underline{\tilde{x}}^{(k)} = (\tilde{x}_1^{(k)},\ldots,\tilde{x}_n^{(k)})_{k\in\mathbb{N}}$ with at least one component $\tilde{x}_i^{(k)}$ which tend to ∞ as k tends to infinity. Use the inverse of the preceding homeomorphism to show that this correspond to a sequence of points $\underline{x}^{(k)} = p^{-1}(\underline{\tilde{x}}^{(k)})$ which tends to the north pole ∞.