REAL ALGEBRAIC GEOMETRY LECTURE NOTES (28: 04/02/10)

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Contents

1. Examples

If G is a Hahn group, namely a Hahn sum

$$G = \coprod_{\gamma\in\Gamma} B(\gamma)$$

or a Hahn product

$$G = \mathrm{H}_{\gamma\in\Gamma}\, B(\gamma)$$

as in section 2 of Lecture 23, then the valued $\mathbb{Q}$-vector space $(G, v_{\min})$ is isomorphic to (G, v), where v is the natural valuation explained in the last lecture (Lecture 27, section 3). Namely

$$\forall\, x, y \in G \qquad v(x) = v(y) \;\Leftrightarrow\; v_{\min}(x) = v_{\min}(y).$$

2. Valued fields

Definition 2.1. Let K be a field, G an ordered abelian group and ∞ an element greater than every element of G. A surjective map

$$w\colon K \;\longrightarrow\; G \cup \{\infty\}$$

is a **valuation** if and only if $\forall\, a, b \in K$:

(i) $w(a) = \infty \;\Leftrightarrow\; a = 0$.

(ii) $w(ab) = w(a) + w(b)$.

(iii) $w(a - b) \geqslant \min\{w(a), w(b)\}$.

Immediate consequences are:

- $w(a) = w(-a)$,
- $w(a^{-1}) = -w(a)$ if $a \neq 0$,
- $w(a) \neq w(b) \;\Rightarrow\; w(a+b) = \min\{w(a), w(b)\}$.

Definition 2.2.

$$R_w := \{a \in K : w(a) \geqslant 0\} \text{ is the } \textbf{valuation ring}.$$
$$I_w := \{a \in K : w(a) > 0\} \text{ is the } \textbf{valuation ideal}.$$

Lemma 2.3. *I_w is an ideal of the ring R_w and it is maximal proper.*

Thus R_w/I_w is a field denoted by K_w and called the **residue field**. The **residue map** is the canonical surjection:

$$\begin{array}{ccc} R_w & \longrightarrow & R_w/I_w \\ b & \mapsto & b + I_w := b_w \end{array}$$

The **group of units** of the valuation ring R_w is given by

$$\mathcal{U}_w = \{a \in K : w(a) = 0\}$$

and it is a subgroup of the multiplicative group of R_w.
The **group of 1-units** is the multiplicative subgroup of $\mathcal{U}_w$ given by

$$1 + I_w = \{a \in K : w(a-1) > 0\}.$$

3. The natural valuation of an ordered field

Let $(K, +, \cdot, 0, 1, <)$ be a totally ordered field.

Remark 3.1. $(K, +, 0, <)$ is a totally ordered divisible abelian group.

So we have the natural valuation v on K as a $\mathbb{Q}$-vector space. Setting $G := v(K \setminus \{0\})$, we have:

$$\begin{array}{rcl} v\colon K & \longrightarrow & G \cup \{\infty\} \\ 0 \neq a & \mapsto & v(a) := [a] \\ 0 & \mapsto & \infty \end{array}$$

We shall show now that we can endow the totally ordered value set $(G, <)$ with a group operation $+$ such that $(G, +, <)$ is a totally ordered abelian group. For every $a, b \in K \setminus \{0\}$ define

$$[a] + [b] := [ab].$$

Lemma 3.2. *This addition is well defined and $(G, +, <)$ is a totally ordered abelian group.*

4. The field of power series

Let K be a field and G a totally ordered abelian group.

The field of formal power series with coefficients in K and exponent in G is the set of formal objects

$$K((G)) := \{s = \sum_{g \in G} s(g)t^g : s(g) \in K \text{ and } \operatorname{support}(s) = \{g \in G : s(g) \neq 0\}$$
$$\text{is well ordered in G}\}$$

with the following addition and multiplication:

$$(\sum_{g \in G} s(g)t^g) + (\sum_{g \in G} r(g)t^g) := \sum_{g \in G} (s(g) + r(g))\, t^g.$$

$$(\sum_{g \in G} s(g)t^g) \cdot (\sum_{g \in G} r(g)t^g) := \sum_{g \in G} (\sum_{g' \in G} r(g')s(g - g'))\, t^g.$$

Lemma 4.1. *This multiplication is well defined:*

(1) *the sum is finite.*
(2) $\operatorname{support}(rs)$ *is well ordered.*

To see that $K((G))$ is a field, we compute the inversion function. Let $s \in K((G))$ with $\min \operatorname{support}(s) = g_0$. We can write

$$s = s(g_0)t^{g_0}(1 + \varepsilon),$$

and then

$$s^{-1} = \frac{1}{s(g_0)} t^{-g_0}(1 + \varepsilon)^{-1},$$

with

$$(1 + \varepsilon)^{-1} = \sum_{i \in \mathbb{N}} a_i \varepsilon^i.$$

Example 4.2. If $G = \mathbb{Z}$ and $K = \mathbb{R}$, $K((G)) = \mathbb{R}((\mathbb{Z}))$ is the field of Laurent series with coefficients in $\mathbb{R}$:

$$s = \sum_{n=-m}^{\infty} s(n)t^n \qquad s(n) \in \mathbb{R}.$$