REAL ALGEBRAIC GEOMETRY LECTURE NOTES (27: 02/02/10)

SALMA KUHLMANN

Contents

1. Ordered abelian groups

Definition 1.1. $(G, +, 0, <)$ is an **ordered abelian group** if $(G, +, 0)$ is an abelian group and $<$ is a total order on G such that for every $a, b, c \in G$

$$a \leqslant b \ \Rightarrow \ a + c \leqslant b + c.$$

Definition 1.2. A subgroup C of an ordered abelian group G is **convex** if $\forall\, c_1, c_2 \in C$ and $\forall\, x \in G$

$$c_1 < x < c_2 \ \Rightarrow \ x \in C.$$

Examples 1.3. $C = \{0\}$ and $C = G$ are convex subgroups.

Definition 1.4. Let G be an abelian ordered group, $x \in G$, $x \neq 0$.
We define:

$$C_x := \bigcap \{C : C \text{ is a convex subgroup of } G \text{ and } x \in C\}.$$
$$D_x := \bigcup \{D : D \text{ is a convex subgroup of } G \text{ and } x \notin D\}.$$

A convex subgroup C of G is said to be **principal** if there is some $x \in G$ such that $C = C_x$.

Proposition 1.5.

(1) *D_x is a proper convex subgroup of C_x.*

(2) *D_x is the largest proper convex subgroup of C_x, i.e. if C is a convex subgroup such that*

$$D_x \subseteq C \subseteq C_x$$

then $C = D_x$ or $C = C_x$.

(3) *It follows that the ordered abelian group C_x/D_x has no non-trivial proper convex subgroup.*

2. Archimedean groups

Definition 2.1. Let $(A, +, 0, <)$ be an ordered abelian group. We say that A is **archimedean** if for all non-zero $a_1, a_2 \in A$:

$$\exists\, n \in \mathbb{N}: \quad n|a_1| > |a_2| \ \text{ and } \ n|a_2| > |a_1|,$$

where for every $a \in A$, $|a| := \max\{a, -a\}$.

Proposition 2.2. *(Hölder) Every archimedean group is isomorphic to a subgroup of* $(\mathbb{R}, +, 0, <)$.

Proposition 2.3. *A is archimedean if and only if A has no non-trivial proper convex subgroup.*

Therefore if G is an ordered group and $x \in G$ with $x \neq 0$, the quotient C_x/D_x is archimedean (by 2.3) and can be embedded in $(\mathbb{R}, +, 0, <)$ (by 2.2).

Definition 2.4. Let G be an ordered group, $x \in G$, $x \neq 0$. We say that

$$B_x := C_x/D_x$$

is **the archimedean component** of x in G.

3. Archimedean equivalence

Definition 3.1. An abelian group G is **divisible** if for every $x \in G$ and for every $n \in \mathbb{N}$ there is $y \in G$ such that $x = ny$.

Remark 3.2. Any ordered divisible abelian group G is a $\mathbb{Q}$-vector space and G can be viewed as a valued $\mathbb{Q}$-vector space in a natural way.

Definition 3.3. (archimedean equivalence) For every $x, y \in G$ we define

$$x \ \sim^+ \ y \quad \Leftrightarrow \quad \exists\, n \in \mathbb{N} \quad n|x| \geqslant |y| \ \text{ and } \ n|y| \geqslant |x|.$$

$$x \ <<^+ \ y \quad \Leftrightarrow \quad \forall\, n \in \mathbb{N} \quad n|x| < |y|.$$

Proposition 3.4.

(1) $\sim^+$ *is an equivalence relation.*

(2) $\sim^+$ *is compatible with* $<<^+$*:*

$$\begin{array}{lllll} x <<^+ y & \text{and} & x \sim^+ z & \Rightarrow & z <<^+ y, \\ x <<^+ y & \text{and} & y \sim^+ z & \Rightarrow & x <<^+ z. \end{array}$$

Because of the last proposition we can define an order $<_\Gamma$ on $\Gamma := G/\sim^+ = \{[x] : x \in G\}$ as follows:

$$[y] <_\Gamma [x] \quad \Leftrightarrow \quad x <<^+ y.$$

Proposition 3.5.

(1) Γ *is a totally ordered set under* $<_\Gamma$.

(2) *The map*

$$\begin{array}{rcll} v\colon G & \longrightarrow & \Gamma \cup \{\infty\} & \\ 0 & \mapsto & \infty & \\ x & \mapsto & [x] & (\text{if } x \neq 0) \end{array}$$

is a valuation on G *as a* $\mathbb{Z}$*-module:*

For every $x, y \in G$*:*

- $v(x) = \infty$ *iff* $x = 0$,
- $v(nx) = v(x) \quad \forall\, n \in \mathbb{Z},\ n \neq 0$,
- $v(x + y) \geqslant \min\{v(x), v(y)\}$.

(3) *if* $x \in G$, $x \neq 0$, $v(x) = \gamma$, *then*

$$\begin{array}{l} G^\gamma := \{a \in G : v(a) \geqslant \gamma\} = C_x. \\ G_\gamma := \{a \in G : v(a) > \gamma\} = D_x. \end{array}$$

So

$$B_x = C_x/D_x = G^\gamma/G_\gamma = B(\gamma)$$

is the archimedean component associated to γ.