REAL ALGEBRAIC GEOMETRY LECTURE NOTES (22: 14/01/10)

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Contents

Let R be a real closed field.

1. Semialgebraic dimension

Theorem 1.1. *Let $S \subset R^n$ be a semialgebraic set and $T_1, \ldots, T_q$ finitely many semialgebraic subsets of S. Then*

$$S = \dot{\bigcup_{k=1,\ldots,r}} \Sigma_k, \qquad \text{where}$$

(i) *every Σ_k is semialgebraic homeomorphic to an open hypercube $(0,1)^{d_k}$;*

(ii) *the closure of Σ_k in S is the union of Σ_k and some Σ_j with $j \neq k$ and $d_j < d_k$;*

(iii) *the closure $\bar{\Sigma}_k$ of Σ_k is the union of Σ_k and finitely many semialgebraic sets S_i semialgebraic homeomorphic to an open hypercube $(0,1)^{d_i}$, with $d_i < d_k$;*

(iv) *every T_i is the union of some Σ_k.*

*Such a decomposition $S = \bigcup_k \Sigma_k$ is said to be a **stratification** of S and the $\Sigma_1, \ldots, \Sigma_r$ are called **strata**.*

Proposition 1.2. *Let $S \subset R^n$ be a semialgebraic set. Let*

$$S = \bigcup_{i=1}^{p} C_i \qquad S = \bigcup_{j=1}^{q} D_j$$

be two decompositions of S into a disjoint union of semialgebraic sets, with

$$C_i \quad \text{semialgebraic isomorphic to } (0,1)^{d_i} \quad \forall\, i = 1, \ldots, p,$$

$$D_j \quad \text{semialgebraic isomorphic to } (0,1)^{d_j} \quad \forall\, j = 1, \ldots, q.$$

Then $\max_{i=1,\ldots,p}\{d_i\} = \max_{j=1,\ldots,q}\{d_j\} = d$.

We define the ***dimension*** *of* S *such a* d. *We write* $\dim S = d$.

Proof. We can apply Theorem 1.1 taking the semialgebraic subsets $T_{ij} = C_i \cap D_j$, for $i = 1, \dots, p$ and $j = 1, \dots, q$, and we find a stratification

$$S = \bigcup_{k=1}^{r} \Sigma_k$$

which is a common refinement of the two decomposition, i.e. each C_i and each D_j is a finite union of some Σ_k and each Σ_k is semialgebraic homeomorphic to $(0,1)^{d_k}$.

We want to show that $\max_{i=1,\dots,p}\{d_i\} = \max_{j=1,\dots,q}\{d_j\} = \max_{k=1,\dots,r}\{d_k\}$.

Set $\bar{d}_i := \max_{i=1,\dots,p}\{d_i\}$ and $\bar{d}_k := \max_{k=1,\dots,r}\{d_k\}$.
Since every Σ_k is contained in some C_i, of course $\bar{d}_k \leqslant \bar{d}_i$.

Let now Σ_k a stratum semialgebraic homeomorphic to $(0,1)^{\bar{d}_k}$ and suppose that $\Sigma_k \subset C_i$. We claim that Σ_k is open in C_i (equivalently, $C_i \setminus \Sigma_k$ is closed in C_i): by Theorem 1.1(ii), if Σ_s is a stratum in $C_i \setminus \Sigma_k$ then the closure of Σ_s in C_i contains only Σ_s and strata Σ_a with $d_a < d_s \leqslant \bar{d}_k$. Therefore the closure of $C_i \setminus \Sigma_k$ in C_i is disjoint from Σ_k and this shows that $C_i \setminus \Sigma_k$ is closed in C_i (and Σ_k is open in C_i). We conclude assuming the following fact:

Fact 1.3.

- $A \subset X$, X homeomorphic to $(0,1)^d$, A open in X $\Rightarrow$ A locally homeomorphic to $(0,1)^d$ (i.e. for every $x \in A$ there is an open neighborhood of x homeomorphic to $(0,1)^d$).
- $(0,1)^{d_1}$ is homeomorphic to $(0,1)^{d_2}$ $\Leftrightarrow$ $d_1 = d_2$.

Therefore $\bar{d}_k = \bar{d}_i$, and $\bar{d}_k = \bar{d}_j$ is similar. □

Remark 1.4. Let A, $B \subset R^n$ be semialgebraic sets. Then

(1) $\dim(A \cup B) = \max\{\dim A, \dim B\}$.

(2) $\dim(A \times B) = \dim A + \dim B$.

We see now that the dimension of a semialgebraic set behaves well with respect to the topological closure:

Proposition 1.5. *Let* $S \subset R^n$ *be semialgebraic. Then*

(i) $\dim \bar{S} = \dim S$.

(ii) $\dim(\bar{S} \setminus S) < \dim S$.

Proof. Let us observe that by 1.4(1), (ii) $\Rightarrow$ (i).

We claim that if

$$S = \dot{\bigcup_{k=1,\dots,r}} \Sigma_k$$

is a stratification of S as in Theorem 1.1, then

$$\bar{S} = \bigcup_{k=1}^{r} \bar{\Sigma}_k :$$

($\subseteq$) $\bigcup_{k=1}^{r} \bar{\Sigma}_k$ is a finite union of closed set, so it is closed. It contains S, so it contains also the closure $\bar{S}$ of S.

($\supseteq$) For every $k = 1, \dots, r$, $\Sigma_k \subseteq S$. Then $\bar{\Sigma}_k \subseteq \bar{S}$ and $\bigcup_{k=1}^{r} \bar{\Sigma}_k \subseteq \bar{S}$.

Therefore $\dim(\bar{S} \setminus S) \leqslant \max\{\dim(\bar{\Sigma}_k \setminus \Sigma_k) : 1 \leqslant k \leqslant r\}$ and by Theorem 1.1(iii) this is stricly less than $\max\{\dim \Sigma_k : 1 \leqslant k \leqslant r\} = \dim S$. □

Now we see that the dimension of a semialgebraic set is invariant by semialgebraic bijections (not necessarily continuous!):

Lemma 1.6. *Let $A \subset R^{n+k}$ be a semialgebraic set, $\pi \colon R^{n+k} \to R^n$ the projection on the first n coordinates. Then $\dim \pi(A) \leqslant \dim A$. Moreover if $\pi_{|A} \colon A \to R^n$ is injective, then $\dim \pi(A) = \dim A$.*

Proof. By induction on k.

- $k = 1$. Write A as a disjoint union of cells.

- $k \Rightarrow k+1$. Consider the projection $\pi \colon R^{n+k+1} \to R^n$ on the first n coordinates as the composition of the projection $\pi_1 \colon R^{n+k+1} \to R^{n+1}$ on the first $n+1$ coordinates and the projection $\pi_2 \colon R^{n+1} \to R^n$ on the first n coordinates:

$$\begin{tikzcd} R^{n+1+k} \arrow[r, "\pi_1"] \arrow[rr, bend left, "\pi"] & R^{n+1} \arrow[r, "\pi_2"] & R^n \\ A & A_1 & \pi(A) \end{tikzcd}$$
$$A \mapsto A_1 \mapsto \pi(A)$$

Then by induction $\dim A \geqslant \dim \pi_1(A) = A_1 \geqslant \dim \pi_2(A_1) = \pi(A)$.

Moreover

$$\pi_{|A} \text{ is injective} \iff \pi_{1|A} \text{ and } \pi_{2|A_1} \text{ are injective.}$$

□

Theorem 1.7. *Let $S \subset R^n$ be semialgebraic, $f \colon S \to R^k$ a semialgebraic map (not necessarily continuous). Then $\dim f(S) \leqslant \dim S$. If f is injective then $\dim f(S) = \dim S$.*

Proof. Let $A \subset R^{n+k}$ be the graph of f:

$$A = \Gamma(f) = \{(\underline{x}, f(\underline{x})) : \underline{x} \in S\}.$$

Let $\pi_1 \colon R^{n+k} \to R^n$ be the projection on the first n coordinates. Then $\pi_{1|A}$ is injective and $\pi_1(A) = S$. Therefore, by Lemma 1.6, $\dim S = \dim A$.

Let now $\pi_2 \colon R^{n+k} \to R^k$ be the projection on the last k coordinates. Then $\pi_2(A) = f(S)$. Again by Lemma 1.6 $\dim f(S) \leqslant \dim A = \dim S$.

If f is injective then $\dim f(S) = \dim A$. □

2. Algebraic dimension

Consider the ring of polynomials $R[\underline{\mathrm{x}}] := R[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ in n variables and coefficients in R.

An algebraic set $V \subset R^n$ is by definition the common zeroset of all polynomials belonging to a subset $A \subset R[\underline{\mathrm{x}}]$:

$$V = \mathcal{Z}(A) := \{\underline{x} \in R^n : p(\underline{x}) = 0 \; \forall \, p \in A\}.$$

Then we can consider the set of polynomials which vanish on V (which of course contains A):

$$\mathcal{I}(V) := \{p \in R[\underline{\mathrm{x}}] : p(\underline{x}) = 0 \; \forall \, \underline{x} \in V\}.$$

We take the ring of polynomal functions on V, i.e. the quotient of $R[\underline{\mathrm{x}}]$ by $\mathcal{I}(V)$:

$$\mathcal{P}(V) := \frac{R[\underline{\mathrm{x}}]}{\mathcal{I}(V)}.$$

And now we are ready to define the algebraic dimension of V:

Definition 2.1. The **dimension** of an algebraic set V is by definition the Krull dimension of $\mathcal{P}(V)$, i.e. the maximal $d \in \mathbb{N}$ such that

$$\exists \; P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_d,$$

where P_i is a prime ideal of $\mathcal{P}(V) \; \forall \; i = 1, \ldots, d$.

We recall that an ideal P is said to be **prime** if for every pair of ideals A and B,

$$AB \subset P \; \Rightarrow \; A \subset P \text{ or } B \subset P.$$

In general, given a subset $S \subset R^n$, $\mathcal{Z}(\mathcal{I}(S))$ is the smallest algebraic subset of R^n containing S. It is said to be the **Zariski closure** of S and it is denoted by $\bar{S}^Z$.

In fact, the algebraic subsets of R^n are the closed sets of the **Zariski topology**, and $\bar{S}^Z$ is the closure of S with respect to this topology.

The Zariski topology is coarser than the Euclidean topology, i.e. each algebraic set is closed in the Euclidean topology, but the converse is not true.

Theorem 2.2. *Let $S \subset R^n$ be a semialgebraic set. Then its dimension as a semialgebraic set is equal to the dimension, as an algebraic set, of its Zariski closure $\bar{S}^Z$. In particular, if $V \subset R^n$ is an algebraic set, then its dimension as a semialgebraic set is equal to its dimension as an algebraic set (i.e. the Krull dimension of $\mathcal{P}(V)$).*

Dimension will be investigated more during next term.